EXISTENCE/UNIQUENESS AND VALIDATION OF PARAMETER
IMBEDDING EQUATIONS FOR NONLINEAR
FREDHOLM INTEGRAL EQUATIONS

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Technical Report No. 104

April, 1979

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ABSTRACT

A parameter imbedding method for nonlinear Fredholm integral equations produces a Cauchy system involving the solution of the integral equation and an associated resolvent kernel. For such Cauchy systems sufficient conditions are given to guarantee local existence and uniqueness of solutions. A Picard type theorem utilizing a Lipschitz condition is obtained. This result then yields the validation of the Cauchy system.

I. INTRODUCTION

An invariant imbedding technique for solving Fredholm integral equations has been described in [1,2], which parameterizes the interval length. The question of existence and uniqueness of solutions of the resulting Cauchy system is discussed in [3] for the linear case and in [4] for the nonlinear case. The validation of the imbedding method is then considered in [5].

A similar imbedding technique is developed for integral equations involving a parameter by Kagiwada and Kalaba [1, Chapter 8]. We consider solutions of the nonlinear Fredholm integral equation with parameter

$$u(t) = g(t, \lambda) + \lambda \int_0^1 k(t, y, \lambda, u(y)) dy, \quad 0 \leq t \leq 1, \quad (1.1)$$

where the parameter λ may occur naturally in the statement of the problem or may be artificially introduced. To emphasize the dependence of the solution on λ (1.1) is written as

$$u(t, \lambda) = g(t, \lambda) + \lambda \int_0^1 k(t, y, \lambda, u(y, \lambda)) dy, \quad 0 \leq t \leq 1. \quad (1.2)$$

If $K(t, y, \lambda)$ is the resolvent with respect to the linear integral equation

$$w(t, \lambda) = F(t, \lambda) + \lambda \int_0^1 k_u(t, y, \lambda, u(y, \lambda)) w(y, \lambda) dy, \quad 0 \leq t \leq 1,$$

Then $K(t, y, \lambda)$ satisfies the integral equation

$$K(t, y, \lambda) = k_u(t, y, \lambda, u(t, \lambda)) + \lambda \int_0^1 k_u(t, r, \lambda, u(r, \lambda)) K(r, y, \lambda) dr, \quad (1.3)$$

$0 \leq t, y \leq 1, \quad 0 \leq \lambda \leq \Lambda$. The parameter imbedding method in [1] then

leads to the following Cauchy system for the solutions $u(t, \lambda)$ of (1.2) and $K(t, y, \lambda)$ of (1.3):

$$u_\lambda(t, \lambda) = Y(t, \lambda) + \lambda \int_0^1 K(t, r, \lambda) Y(r, \lambda) dr, \quad (1.4i)$$

$$K_\lambda(t, y, \lambda) = Q(t, y, \lambda) + \lambda \int_0^1 K(t, r, \lambda) Q(r, y, \lambda) dr, \quad (1.4ii)$$

$$u(t, 0) = g(t, 0), \quad (1.5i)$$

$$K(t, y, 0) = k_u(t, y, 0, g(y, 0)), \quad (1.5ii)$$

for $0 \leq t, y \leq 1$ and $0 \leq \lambda \leq \Lambda$, where

$$\begin{aligned} Y(t, y) &= g_\lambda(t, \lambda) + \int_0^1 k(t, r, \lambda, u(r, \lambda)) dr \\ &+ \lambda \int_0^1 k_\lambda(t, r, \lambda, u(r, \lambda)) dr, \end{aligned} \quad (1.6)$$

and

$$\begin{aligned} Q(t, y, \lambda) &= k_{u\lambda}(t, y, \lambda, u(y, \lambda)) + k_{uu}(t, y, \lambda, u(y, \lambda)) \\ &\cdot \left\{ Y(y, \lambda) + \lambda \int_0^1 K(y, r, \lambda) Y(r, \lambda) dr \right\} \\ &+ \int_0^1 k_u(t, r, \lambda, u(r, \lambda)) K(r, y, \lambda) dr \\ &+ \lambda \int_0^1 k_{u\lambda}(t, r, \lambda, u(r, \lambda)) K(r, y, \lambda) dr \\ &+ \lambda \int_0^1 k_{uu}(t, r, \lambda, u(r, \lambda)) K(r, y, \lambda) \\ &\cdot \left\{ Y(r, \lambda) + \lambda \int_0^1 K(r, s, \lambda) Y(s, \lambda) ds \right\} dr. \end{aligned} \quad (1.7)$$

A solution of the initial value system (1.4)-(1.5) then affords a numerical technique for computing solutions to the nonlinear integral equation (1.2).

This paper considers a Picard type theorem which under suitable hypothesis will guarantee local existence and uniqueness of solutions of (1.4)-(1.5). This theorem is then applied to the question of validation of the Cauchy system; that is, we show that solutions of the Cauchy system do indeed yield solutions of the original integral equation (1.2). This validation is obtained under the same conditions which produce unique solutions.

II. EXISTENCE AND UNIQUENESS

Let I denote the interval $[0,1]$ and Λ denote the interval $[0,\Lambda]$. Consider solutions in R^n of the integrodifferential system

$$\begin{aligned} \frac{dV}{d\lambda} = & G(t,y,\lambda,U(y,y,\lambda)) \\ & + \int_0^1 H(t,y,r,\lambda,U(t,r,\lambda),U(y,y,\lambda),U(r,y,\lambda),U(y,r,\lambda))dr \\ & + \int_0^1 \int_0^1 J(t,y,r,s,\lambda,U(t,r,\lambda),U(s,y,\lambda),U(y,y,\lambda),U(t,y,\lambda), \\ & \quad U(t,r,\lambda),U(r,y,\lambda),U(r,s,\lambda))ds dr \\ & + \int_0^1 \int_0^1 \int_0^1 T(t,y,r,s,z,\lambda,U(r,y,\lambda),U(r,s,\lambda),U(z,y,\lambda))dz ds dr \end{aligned} \quad (2.1)$$

with initial condition

$$U(t,y,0) = \Phi(t,y), \quad (2.2)$$

where $t,y \in I$ and $\lambda \in \Lambda$. As will be shown in Section 3, the system (1.4)-(1.5) can be expressed in the form (2.1)-(2.2), which is the motivation for considering the above form.

If $\|\cdot\|$ denotes some convenient norm on R^n , then the ball of radius b about the initial value $\Phi(t,y)$ is given by

$$S = \left\{ U \in R^n \mid \|U - \Phi(t, y)\| \leq b, \quad t, y \in I \right\}.$$

The continuity conditions are

$$\Phi \in C[I^2, R^n], \quad (2.3i)$$

$$G \in C[I^2 \times \Lambda \times S, R^n], \quad (2.3ii)$$

$$H \in C[I^3 \times \Lambda \times S^4, R^n], \quad (2.3iii)$$

$$J \in C[I^4 \times \Lambda \times S^7, R^n], \quad (2.3iv)$$

$$T \in C[I^5 \times \Lambda \times S^3, R^n]. \quad (2.3v)$$

The Lipschitz condition is given by

$$\|G(t, y, \lambda, U) - G(t, y, \lambda, V)\| \leq L\|U - V\| \quad (2.4i)$$

for all (t, y, λ, U) and $(t, y, \lambda, V) \in I^2 \times \Lambda \times S$,

$$\begin{aligned} & \|H(t, y, r, \lambda, U_1, \dots, U_4) - H(t, y, r, \lambda, V_1, \dots, V_4)\| \\ & \leq L(\|U_1 - V_1\| + \dots + \|U_4 - V_4\|) \end{aligned} \quad (2.4ii)$$

for all $(t, y, r, \lambda, U_1, \dots, U_4)$ and $(t, y, r, \lambda, V_1, \dots, V_4) \in I^3 \times \Lambda \times S^4$,

$$\begin{aligned} & \|J(t, y, r, s, \lambda, U_1, \dots, U_7) - J(t, y, r, s, \lambda, V_1, \dots, V_7)\| \\ & \leq L(\|U_1 - V_1\| + \dots + \|U_7 - V_7\|), \end{aligned} \quad (2.4iii)$$

for all $(t, y, r, s, \lambda, U_1, \dots, U_7)$ and $(t, y, r, s, \lambda, V_1, \dots, V_7) \in I^4 \times \Lambda \times S^7$,

$$\begin{aligned} & \|T(t, y, r, s, z, \lambda, U_1, U_2, U_3) - T(t, y, r, s, z, \lambda, V_1, V_2, V_3)\| \\ & \leq L(\|U_1 - V_1\| + \|U_2 - V_2\| + \|U_3 - V_3\|), \end{aligned} \quad (2.4iv)$$

for all $(t, y, r, s, z, \lambda, U_1, U_2, U_3)$ and $(t, y, r, s, z, \lambda, V_1, V_2, V_3) \in I^5 \times \Lambda \times S^3$.

Let

$$M_1 = \sup \left\{ \|G(t, y, \lambda, U)\| \mid (t, y, \lambda, U) \in I^2 \times \Lambda \times S \right\},$$

$$M_2 = \sup \left\{ \|H(t, y, r, \lambda, U_1, \dots, U_4)\| \mid (t, y, r, \lambda, U_1, \dots, U_4) \in I^3 \times \Lambda \times S^4 \right\},$$

$$M_3 = \sup \left\{ \|J(t, y, r, s, \lambda, U_1, \dots, U_7)\| \mid (t, y, r, s, \lambda, U_1, \dots, U_7) \in I^4 \times \Lambda \times S^7 \right\},$$

$$M_4 = \sup \left\{ \|T(t, y, r, s, z, \lambda, U_1, U_2, U_3)\| \mid (t, y, r, s, z, \lambda, U_1, U_2, U_3) \in I^5 \times \Lambda \times S^3 \right\},$$

and define $M = M_1 + M_2 + M_3 + M_4$.

The following theorem gives a sufficient condition for local existence and uniqueness of solutions of (2.1)-(2.2).

Theorem 2.1. Assume the continuity condition (2.3) and the Lipschitz condition (2.4) hold. Let $\Lambda = b/M$. Then there exists a unique solution of (2.1)-(2.2) defined for $(t, y, \lambda) \in I^2 \times \Lambda$.

Remark. The proof of Theorem 2.1 is similar to the proof of Theorem 2.1 in [4]. For this reason and also because of notational problems some details of the proof are omitted.

Proof. The initial value problem (2.1)-(2.2) is equivalent to the integral equation

$$\begin{aligned} U(t, y, \lambda) = & \Phi(t, y) + \int_0^\lambda G(t, y, w, U(y, y, w)) dw \\ & + \int_0^\lambda \int_0^1 H(t, y, r, w, U(t, r, w), U(y, y, w), U(r, y, w), U(y, r, w)) dr dw \\ & + \int_0^\lambda \int_0^1 \int_0^1 J(t, y, r, s, w, U(t, r, w), U(s, y, w), U(y, y, w), U(t, y, w), U(y, r, w), \\ & \quad U(r, y, w), U(r, s, w)) ds dr dw \\ & + \int_0^\lambda \int_0^1 \int_0^1 \int_0^1 T(t, y, r, s, z, w, U(r, y, w), U(r, s, w), U(z, y, w)) dz ds dr dw. \end{aligned} \quad (2.5)$$

According to the method of successive approximations, consider the sequence $\{U_n(t, y, \lambda)\}$ generated by taking $U_0(t, y, \lambda) = \Phi(t, y)$, $(t, y, \lambda) \in I^2 \times \Lambda$, and $U_{n+1}(t, y, \lambda)$ is the right hand side of (2.5) with U replaced by U_n . Clearly, $U_0(t, y, \lambda) \in S$ and is continuous on $I^2 \times \Lambda$ by (2.3i).

If U_n is continuous and $U_n \in S$, then $U_{n+1} \in C[I^2 \times \Lambda, R^n]$ and

$$\begin{aligned} \|U_{n+1}(t, y, \lambda) - \Phi(t, y)\| &\leq \int_0^\lambda M_1 d\omega + \int_0^\lambda M_2 d\omega + \int_0^\lambda M_3 d\omega + \int_0^\lambda M_4 d\omega \\ &= M\lambda \leq b. \end{aligned}$$

Therefore, the sequence of iterates are continuous and stay within the ball S .

In order to show convergence of the sequence of iterates we need an appropriate estimate on the difference of successive iterates. The induction hypothesis is

$$\|U_n(t, y, \lambda) - U_{n-1}(t, y, \lambda)\| \leq M(15L)^{n-1} \frac{\lambda^n}{n!} \leq M(15L)^{n-1} \frac{\Lambda^n}{n!}. \quad (2.6)$$

For $n = 1$ the statement (2.6) follows from the estimate

$$\|U_1(t, y, \lambda) - \Phi(t, y)\| \leq \int_0^\lambda M_1 d\omega + \dots + \int_0^\lambda M_4 d\omega = M\lambda \leq M\Lambda.$$

Considering the general induction step

$$\begin{aligned}
\|U_{n+1}(t, y, \lambda) - U_n(t, y, \lambda)\| &\leq \int_0^\lambda LM(15L)^{n-1} \frac{w^n}{n!} dw \\
&+ \int_0^\lambda \int_0^1 4LM(15L)^{n-1} \frac{w^n}{n!} dr dw + \int_0^\lambda \int_0^1 \int_0^1 7LM(15L)^{n-1} \frac{w^n}{n!} ds dr dw \\
&+ \int_0^\lambda \int_0^1 \int_0^1 \int_0^1 3LM(15L)^{n-1} \frac{w^n}{n!} ds ds dr dw \\
&\leq M(15L)^n \frac{\lambda^{n+1}}{(n+1)!} \leq M(15L)^n \frac{\Lambda^{n+1}}{(n+1)!}.
\end{aligned} \tag{2.7}$$

Thus the estimate (2.6) holds for all $n \geq 1$. The uniform convergence of $U_n(t, y, \lambda)$ on $I^2 \times \Lambda$ follows by comparison with series

$$\sum_{n=1}^{\infty} M(15L)^n \frac{\Lambda^{n+1}}{(n+1)!}.$$

Let $U(t, y, \lambda) = \lim_{n \rightarrow \infty} U_n(t, y, \lambda)$. The Lipschitz condition (2.4) and the uniform convergence of $U_n(t, y, \lambda)$ imply the following:

$$\lim_{n \rightarrow \infty} \|G(t, y, w, U_n(y, y, w)) - G(t, y, w, U(y, y, w))\| = 0,$$

$$\lim_{n \rightarrow \infty} \|H(t, y, r, w, U_n(t, r, w), \dots, U_n(y, r, w))$$

$$- H(t, y, r, w, U(t, r, w), \dots, U(y, r, w))\| = 0,$$

$$\lim_{n \rightarrow \infty} \|J(t, y, r, s, w, U_n(t, r, w), \dots, U_n(r, s, w))$$

$$- J(t, y, r, s, w, U(t, r, w), \dots, U(r, s, w))\| = 0,$$

$$\lim_{n \rightarrow \infty} \|T(t, y, r, s, z, w, U_n(r, y, w), U_n(r, s, w), U_n(z, y, w))$$

$$- T(t, y, r, s, z, w, U(r, y, w), U(r, s, w), U(z, y, w))\| = 0,$$

where all of the above limits are uniform with respect to $t, y, r, s, s \in I$ and $w \in \Lambda$. Thus taking the limit $U_{n+1}(t, y, \lambda)$ in the integral equation defining U_{n+1} shows that $U(t, y, \lambda)$ is a solution of (2.5) and hence of (2.1)-(2.2). This completes the proof of local existence.

Suppose that $V(t, y, \lambda)$ is also a solution of (2.5); then an induction argument using the Lipschitz condition (2.4) gives the estimate

$$\|V(t, y, \lambda) - U_n(t, y, \lambda)\| \leq b(15L)^n \frac{\lambda^n}{n!} \leq b(15L)^n \frac{\Lambda^n}{n!},$$

for $t, y \in I$ and $\lambda \in \Lambda$. Therefore, $\lim_{n \rightarrow \infty} U_n(t, y, \lambda) = V(t, y, \lambda)$, which establishes the uniqueness of solutions. This completes the proof of Theorem 2.1.

Remark. Since the solution $U(t, y, \lambda)$ is the uniform limit of continuous functions, the solution is continuous on $I^2 \times \Lambda$.

III. APPLICATION

Existence and uniqueness of solutions of the Cauchy system (1.4)-(1.5) can now be obtained as an application of Theorem 2.1. If the vector function $U(t, y, \lambda)$ is defined by $u_1(t, y, \lambda) = u(t, \lambda)$ and $u_2(t, y, \lambda) = K(t, y, \lambda)$, then the system (1.4)-(1.5) can be expressed as

$$\begin{aligned} \frac{du_1}{d\lambda}(t, y, \lambda) &= g_\lambda(t, \lambda) + \int_0^1 \left\{ k(t, r, \lambda, u_1(r, \lambda)) + \lambda k_\lambda(t, r, \lambda, u_1(r, \lambda)) \right. \\ &\quad \left. + \lambda u_2(t, r, \lambda) g_\lambda(r, \lambda) \right\} dr \\ &+ \int_0^1 \int_0^1 \left\{ \lambda u_2(t, r, \lambda) k(r, s, \lambda, u_1(s, \lambda)) \right. \\ &\quad \left. + \lambda^2 u_2(t, r, \lambda) k_\lambda(r, s, \lambda, u_1(s, \lambda)) \right\} ds dr, \end{aligned} \quad (3.1i)$$

$$\begin{aligned}
\frac{du_2}{d\lambda}(t, y, \lambda) &= k_{u\lambda}(t, y, \lambda, u_1(y, \lambda)) + k_{uu}(t, y, \lambda, u_1(y, \lambda))g_\lambda(t, \lambda) \\
&+ \int_0^1 \left\{ k_{uu}(t, y, \lambda, u_1(y, \lambda)) \left[k(y, r, \lambda, u_1(r, \lambda)) \right. \right. \\
&\quad \left. \left. + \lambda k_\lambda(y, r, \lambda, u_1(r, \lambda)) + \lambda u_2(y, r, \lambda) g_\lambda(r, \lambda) \right] \right. \\
&\quad \left. + k_u(t, r, \lambda, u_1(r, \lambda)) u_2(r, y, \lambda) + \lambda k_{u\lambda}(t, r, \lambda, u_1(r, \lambda)) u_2(r, y, \lambda) \right. \\
&\quad \left. + \lambda k_{uu}(t, r, \lambda, u_1(r, \lambda)) u_2(r, y, \lambda) g_\lambda(r, \lambda) \right\} dr \\
&+ \int_0^1 \int_0^1 \left\{ \lambda k_{uu}(t, y, \lambda, u_1(y, \lambda)) u_2(y, r, \lambda) \left[k(r, s, \lambda, u_1(s, \lambda)) \right. \right. \\
&\quad \left. \left. + \lambda k_\lambda(r, s, \lambda, u_1(s, \lambda)) \right] \right. \\
&\quad \left. + \lambda k_{uu}(t, r, \lambda, u_1(r, \lambda)) u_2(r, y, \lambda) \left[k(r, s, \lambda, u_1(s, \lambda)) \right. \right. \\
&\quad \left. \left. + \lambda k_\lambda(r, s, \lambda, u_1(s, \lambda)) + \lambda u_2(r, s, \lambda) g_\lambda(s, \lambda) \right] \right\} ds dr \\
&+ \int_0^1 \int_0^1 \int_0^1 \lambda^2 k_{uu}(t, r, \lambda, u_1(r, \lambda)) u_2(r, y, \lambda) u_2(r, s, \lambda) \\
&\quad \cdot \left\{ k(s, z, \lambda, u_1(z, \lambda)) + \lambda k_\lambda(s, z, \lambda, u_1(z, \lambda)) \right\} dz ds dr. \quad (3.1ii)
\end{aligned}$$

Initially

$$u_1(t, 0) = g(t, 0), \quad (3.2i)$$

$$u_2(t, y, 0) = k_u(t, y, 0, g(y, 0)), \quad (3.2ii)$$

for all $t, y \in I$ and $\lambda \in \Lambda$.

In vector form the system (3.1)-(3.2) is observed to be of the form (2.1)-(2.2). The continuity condition (2.3) is satisfied when it is assumed that

$$g, g_\lambda \in C[I \times \Lambda, R], \quad (3.3i)$$

$$k, k_\lambda, k_{u\lambda}, k_u, k_{uu} \in C[I^2 \times \Lambda \times S, R]. \quad (3.3ii)$$

The Lipschitz condition is given by

$$|\tilde{k}(t, y, \lambda, u) - \tilde{k}(t, y, \lambda, \bar{u})| \leq L|u - \bar{u}|, \quad (3.4)$$

whenever $t, y \in I$, $\lambda \in \Lambda$, $u, \bar{u} \in S$ and $\tilde{k}$ denotes any of the functions $k, k_\lambda, k_u, k_{u\lambda}, k_{uu}$. It is necessary to show that the Lipschitz condition (3.4) implies the Lipschitz condition (2.4). Let $(t, y, \lambda) \in I^2 \times \Lambda$ and $U, \bar{U} \in S$, then

$$\begin{aligned} \|G(t, y, \lambda, U) - G(t, y, \lambda, \bar{U})\| &\leq |k_{u\lambda}(t, y, \lambda, u_1) + k_{uu}(t, y, \lambda, u_1)g_\lambda(t, \lambda) \\ &\quad - k_{u\lambda}(t, y, \lambda, \bar{u}_1) - k_{uu}(t, y, \lambda, \bar{u}_1)g_\lambda(t, \lambda)| \\ &\leq L|u_1 - \bar{u}_1| + L|u_1 - \bar{u}_1| \\ &\leq L\|U - \bar{U}\|, \end{aligned}$$

where we have used the continuity of g_λ and the compactness of I and Λ . The proof that H, J , and T are Lipschitzian is similar to the above but very lengthy and is therefore omitted.

In summary, if $g, g_\lambda, k, k_\lambda, k_u, k_{u\lambda}, k_{uu}$ satisfy the continuity condition (3.3) and the Lipschitz condition (3.4), an application of Theorem 2.1 yields the local existence and uniqueness of solutions of the system (1.4)-(1.5).

IV. VALIDATION

The problem of validation is to show that solutions of the imbedding equations do produce solutions of the original integral equation. It is assumed that $u(t, \lambda)$ and $K(t, y, \lambda)$ are solutions of the system (1.4)-(1.5). It is necessary to show that $K(t, y, \lambda)$ is a solution of (1.3) and $u(t, \lambda)$ is a solution of (1.2). Define the function $A(t, y, \lambda)$ to be the right hand side of equation (1.3), that is

$$A(t, y, \lambda) = k_u(t, y, \lambda, u(y, \lambda)) + \lambda \int_0^1 k_u(t, r, \lambda, u(r, \lambda)) K(r, y, \lambda) dr,$$

$t, y, \lambda \in I^2 \times \Lambda$. As shown in [1] $A(t, y, \lambda)$ satisfies the differential equation

$$A_\lambda(t, y, \lambda) = Q(t, y, \lambda) + \lambda \int_0^1 A(t, r, \lambda) Q(r, y, \lambda) dr \quad (4.1)$$

with initial condition

$$A(t, y, 0) = k_u(t, y, 0, g(y, 0)), \quad (t, y, \lambda) \in I^2 \times \Lambda, \quad (4.2)$$

where Q is given by (1.7)

It should be remarked that in equation (4.1) the function $Q(t, y, \lambda)$ is dependent on the given solutions $u(t, \lambda)$ and $K(t, y, \lambda)$ and is independent of $A(t, y, \lambda)$. It is clear that $K(t, y, \lambda)$ is a solution of (4.1)-(4.2). Therefore, if (4.1)-(4.2) possesses a unique solution, then $A(t, y, \lambda) = K(t, y, \lambda)$. This will establish $K(t, y, \lambda)$ a solution of (1.3).

The system (4.1)-(4.2) is of the form given by (2.1)-(2.2). We make the identifications between equation (2.1) and equation (4.1) as

$$U(t, y, \lambda) = A(t, y, \lambda),$$

$$G(t, y, \lambda, A(y, y, \lambda)) = k_u(t, y, \lambda, u(y, \lambda))$$

$$\begin{aligned} H(t, y, y', \lambda, A(t, y', \lambda), A(y, y, \lambda), A(y', y, \lambda), A(y, y', \lambda)) \\ = \lambda k_u(t, y', \lambda, u(y', \lambda)) K(y', y, \lambda) \end{aligned}$$

$$J \equiv 0,$$

$$T \equiv 0,$$

and between equation (2.2) and (4.2) as

$$\Phi(t, y) = k_u(t, y, 0, g(y, 0)), \quad t, y \in I.$$

The continuity condition (2.3) and the fact that $u(t, \lambda)$ and $K(t, y, \lambda)$ are continuous imply the continuity of the functions G, H, Φ . The Lipschitz condition follows easily from the Lipschitz assumption on k_u . An application of Theorem 2.1 assures a unique solution of (4.1)-(4.2).

It now remains to show that $u(t, \lambda)$ satisfies the integral equation (1.2). Let

$$B(t, \lambda) = g(t, \lambda) + \lambda \int_0^1 k(t, y, \lambda, u(y, \lambda)) dy, \quad 0 \leq t \leq 1, \quad 0 \leq \lambda \leq \Lambda.$$

Differentiation of $B(t, \lambda)$ and equations (1.3) and (1.5) leads to the differential equation,

$$B_\lambda(t, \lambda) = \Psi(t, \lambda) + \lambda \int_0^1 K(t, y, \lambda) \Psi(y, \lambda) dy, \quad 0 \leq t \leq 1, \quad (4.3)$$

with initial condition

$$B(t, 0) = g(t, 0), \quad 0 \leq t \leq 1, \quad (4.4)$$

where Ψ is given by (1.5).

Comparing (4.3) with (1.3) and (4.4) with (1.4) we see that $B_\lambda(t, \lambda) = u_\lambda(t, \lambda)$ and $B(t, 0) = u(t, 0)$. Thus $B(t, \lambda) = u(t, \lambda)$. This completes the validation for the parameter imbedding equations for the nonlinear integral equation.

V. LINEAR INTEGRAL EQUATIONS

Two special cases of the nonlinear integral equation (1.2) is considered in [1, Chapter 8]. The first is the linear integral equation with multiplicative parameter

$$u(t, \lambda) = g(t) + \lambda \int_0^1 k(t, y) u(y, \lambda) dy, \quad 0 \leq t \leq 1, \quad 0 \leq \lambda \leq \Lambda. \quad (5.1)$$

Second is the linear integral equation with nonlinear parameter

$$u(t, \lambda) = g(t) + \lambda \int_0^1 k(t, y, \lambda) u(y, \lambda) dy, \quad 0 \leq t \leq 1, \quad 0 \leq \lambda \leq \Lambda. \quad (5.2)$$

It can be shown in each of the above cases that the Cauchy system related to the nonlinear equation reduces to the Cauchy system related to the respective linear integral equation. Thus the existence and uniqueness of solutions of the Cauchy system related to the above linear equations is guaranteed if the forcing function g and the kernel k is continuous in each of their arguments. The Lipschitz condition for the linear kernel is trivially satisfied. Thus the validation of the Cauchy systems related to equations (5.1) and (5.2) follows from the more general result for the nonlinear case.

VI. SUMMARY

Parameter imbedding has been a successful method in the solution of integral equations. The method replaces the integral equation by a Cauchy system which can be effectively solved by sophisticated integrator techniques. The Theorem on existence and uniqueness together with the validation of this method now establishes this important tool on a rigorous foundation.

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