QUASI-SOLUTIONS AND THEIR ROLE IN THE QUALITATIVE
THEORY OF DIFFERENTIAL EQUATIONS

by

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1. Introduction

In the study of comparison theorems and extremal solutions for systems of ordinary differential equations [5], one usually imposes a condition on the right hand side known as quasi-monotone nondecreasing property. This property is also needed in proving comparison theorems for second order boundary value problems [9] as well as for the initial boundary value problems for parabolic systems [5, 6]. Also, it is well known that in the method of vector Lyapunov functions which provides an effective tool to investigate the stability of Large Scale Systems [1-4], an unpleasant drawback is the requirement of the quasi-monotone property for the comparison systems.

In systems which represent physical situations, we rather often find that this property is not satisfied.

For example, if T denotes the temperature and c the concentration of a combustible substance, a simple model governing the combustion of the material is given by [7]

$$(1.1) \quad \begin{cases} T_t = k_1 \Delta T + Qc \exp(-E/RT) \\ c_t = k_2 \Delta c - c \exp(-E/RT) \end{cases},$$

where Q is the heat of reaction, k_1 and k_2 are diffusion coefficients, E is the activation energy and R the universal gas constant. (1.1) is considered on a bounded domain Ω with boundary conditions $\frac{\partial T}{\partial \nu} = \frac{\partial c}{\partial \nu} = 0$ on $\partial\Omega$ together with initial conditions $T(0, x) = T_0(x)$ and $c(0, x) = c_0(x)$ on Ω . For a discussion of the derivation of the general equations of chemical kinetics, see [8].

In such situations, the notion of quasisolutions suggested in [6] seems very helpful and is expected to play an important role. In this paper, we systematically define various types of quasisolutions and discuss their role in the investigation of qualitative properties of differential equations. The idea of quasisolution now seems to be very natural and in fact, is imbedded in the classical result of Müller which was effectively utilized in [10], [11]

for proving comparison results and existence of extremal solutions for differential equations in a Banach space.

2. Quasi-solutions of initial value problems for ODES.

Consider the system

$$(2.1) \quad x' = f(t, x) \text{ in } J = [0, \alpha] \quad , \quad x(0) = x_0$$

where $f \in C[J \times \mathbb{R}^n; \mathbb{R}^n]$. To define quasi-solutions of (2.1) we fix for each $i \in \{1, \dots, n\}$, two nonnegative integers p_i, q_i such that $p_i + q_i = n-1$ and we split $x \in \mathbb{R}^n$ into $x = (x_i, [x]_{p_i}, \{x\}_{q_i})$. Then (2.1) becomes

$$(2.2) \quad x'_i = f_i(t, x_i, [x]_{p_i}, \{x\}_{q_i}) \text{ for } 1 \leq i \leq n \quad , \quad x(0) = x_0.$$

Since we also need inequalities between vectors, let us note that these will always be understood to hold componentwise. Then we have

Definition 1. Let $a \in C(J; \mathbb{R}^n)$. Then $y \in C^1(J, \mathbb{R}^n)$ is said to be a quasi-solution of (2.1) (w.r. to a) if

$$(2.3) \quad y'_i = f_i(t, y_i, [y]_{p_i}, \{a\}_{q_i}) \text{ for each } i \leq n \quad , \quad y(0) = x_0.$$

y is an upper quasi-solution if

$$y'_i \geq f_i(t, y_i, [y]_{p_i}, \{a\}_{q_i}) \text{ for } 1 \leq i \leq n \quad , \quad y(0) \geq x_0$$

and a lower quasi-solution if both inequalities are reversed. Finally, $v, w \in C^1(J, \mathbb{R}^n)$ with $v(t) \leq w(t)$ for $t \geq 0$ are said to be coupled upper and lower quasi-solutions of (2.1) if for each i

$$\text{and} \quad w'_i \geq f_i(t, w_i, [w]_{p_i}, \{v\}_{q_i}), \quad w_i(0) \geq x_{0i}$$

$$v'_i \leq f_i(t, v_i, [v]_{p_i}, \{w\}_{q_i}), \quad v_i(0) \leq x_{0i}$$

It is then clear that we define quasi-maximal and quasi-minimal solutions of (2.1) (w.r. to a) to be the maximal and minimal solutions of (2.3) if they exist. In the special case $q_i = 0$ for all i quasi-solutions are just solutions, and in case $p_i = 0$ for all i , the quasi-solutions are determined most easily.

Now, f is said to possess the mixed quasi-monotone property (mqmp for short) if, for each i , $f_i(t, [x]_{p_i}, \{x\}_{q_i})$ is monotone nondecreasing in $\{x\}_{p_i}$ and monotone nonincreasing in $\{x\}_{q_i}$. Let us begin by proving the following comparison result.

Theorem 2.1. Let $a \in C(J; \mathbb{R}^n)$; $Q_a = \{(t, x) : t \in J \text{ and } x \geq a(t)\}$; $f \in C(Q_a; \mathbb{R}^n)$ have the mqmp. Let $v \in C(J, \mathbb{R}^n)$, $v(t) \in Q_a(t) = \{x : (t, x) \in Q_a\}$ for all t , $D_-v(t) \leq f(t, v(t))$ and $v(o) \leq x_o$.

Then $v(t) \leq r(t)$ for all t where r , the maximal quasi-solution (w.r. to a), is defined.

Proof. Let $0 < \epsilon \in \mathbb{R}^n$ and x_ϵ a solution of the system (2.3) with f_i replaced by $f_i + \epsilon_i$ and x_o replaced by $x_o + \epsilon$. Since $\lim_{\epsilon \rightarrow 0} x_\epsilon(t) = r(t)$ uniformly on some interval $[o, \beta]$, it is enough to prove $v(t) < x_\epsilon(t)$ in $[o, \beta]$. If this is not true, there exists a first $t_o > o$ such that $v(t) < x_\epsilon(t)$ in $[o, t_o)$, $v(t_o) \leq x_\epsilon(t_o)$ and $v_j(t_o) = x_{\epsilon_j}(t_o)$ for some $j \leq n$. Consequently, $D_-v_j(t_o) \geq x'_{\epsilon_j}(t_o)$ which yields, using the mqmp,

$$\begin{aligned} f_j(t_o, v_j(t_o), [v(t_o)]_{p_j}, \{a(t_o)\}_{q_j}) &\geq f_j(\dots, \{v(t_o)\}_{q_j}) \geq D_-v_j(t_o) \\ &\geq x_{\epsilon_j}(t_o) \geq f_j(t_o, v_j(t_o), [v(t_o)]_{p_j}, \{a(t_o)\}_{q_j}) + \epsilon_j. \end{aligned}$$

This contradiction proves the stated result.

Similarly one can prove the dual result

Theorem 2.2: Let $b \in C(J, \mathbb{R}^n)$, $P_b = \{(t, x) : t \in J \text{ and } x \leq b(t)\}$; $f \in C(P_b, \mathbb{R}^n)$ have the mqmp. Let $v \in C(J, \mathbb{R}^n)$, $v(t) \in P_b(t)$ for all t , $D_-v(t) \geq f(t, v(t))$ and $v(o) \geq x_o$. Then $v(t) \geq r(t)$, where r is the quasi-minimal solution (w.r. to b).

Relative to quasi-upper and lower solutions, we have the following results.

Theorem 2.3. Suppose that the assumptions of Theorem 2.1 hold. Let r be an upper quasi-solution of (2.1) (w.r. to a). Suppose also that f satisfies a "weakened Lipschitz condition", i.e. for each i :

$f_i(t, x_i, [x]_{p_i}, \{a\}_{q_i}) - f_i(t, y_i, [y]_{p_i}, \{a\}_{q_i}) \leq L[x_i - y_i + \sum_{j \in \sigma_i}^{p_i} (x_j - y_j)]$
 provided $x_i \geq y_i$ and $[x]_{p_i} \geq [y]_{p_i}$, where σ_i is the set of indices
 occurring in $[]_{p_i}$.

Then $v(t) \leq r(t)$ in $t \geq 0$.

Proof. Consider $w(t) = r(t) + \epsilon e_0 e^{\lambda t}$, where $\epsilon > 0$, $e_0 = (1, \dots, 1) \in \mathbb{R}^n$
 and $\lambda > Ln$. Then, we have for each i

$$\begin{aligned} w'_i(t) &\geq f_i(t, r_i, [r]_{p_i}, \{a\}_{q_i}) + \epsilon \lambda e^{\lambda t} \geq f_i(t, w_i, [w]_{p_i}, \{a\}_{q_i}) + \\ &\quad + \epsilon \lambda e^{\lambda t} - Ln \epsilon e^{\lambda t} \\ &> f_i(t, w_i, [w]_{p_i}, \{a\}_{q_i}), \end{aligned}$$

$D_- v_i(t) \leq f_i(t, v_i, [v]_{p_i}, \{a\}_{q_i})$ and $v(0) < w(0)$. Hence, $v(t) < w(t)$ by a
 standard argument, and letting $\epsilon \rightarrow 0$, we obtain the desired result.

Theorem 2.4. Let $f \in C(J, \mathbb{R}^n)$ have the mqmp and v, w be coupled upper and
 lower quasi-solutions to (2.1). Suppose that f satisfies a Lipschitz condition
 in the neighborhood of $\bigcup_{t \in J} \{x : v(t) \leq x \leq w(t)\}$. Then $v(t) \leq x(t) \leq w(t)$
 in J for every solution x of (2.1).

Proof. Consider $\tilde{w} = w + \epsilon e_0 e^{\lambda t}$ and $\tilde{v} = v - \epsilon e_0 e^{\lambda t}$ with $\lambda > Ln$, L the Lipschitz
 constant of f , and $\epsilon > 0$ sufficiently small. Then, as in the proof to
 Theorem 2.3, it follows that $\tilde{v}, \tilde{w}$ satisfy the corresponding strict coupled
 inequalities, and therefore $\tilde{v}(t) < x(t) < \tilde{w}(t)$. Now, $\epsilon \rightarrow 0$ gives the result.

We shall next indicate an application of Theorem 2.1. to large scale dynamical
 systems. Let us consider the overall large scale system of the form

$$(2.4) \quad x' = F(t, x), \quad x(t_0) = x_0$$

which is assumed to consist of isolated subsystems

$$(2.5) \quad z'_i = g_i(t, z_i) + R_i(t, x), \quad z_i(t_0) = z_{i0},$$

where $x \in \mathbb{R}^N$, $z_i \in \mathbb{R}^{N_i}$ such that $\sum_{i=1}^n N_i = N$, and the R_i are the interaction
 terms, usually of the form $R_i(t, x) = \sum_{j=1}^n R_{ij}(t, z_j)$. If there
 exist functions V_i, d_i and w_i such that

$$(2.6) \quad D^+V_1(t, z_1) \leq -d_1(t, z_1) + w_1(t, z_1, x)$$

and if the right hand side of (2.6) can be majorized such that (2.6) takes the general form

$$(2.7) \quad D^+V(t, z) \leq f(t, V(t, z))$$

where $f_j(t, z)$ is nondecreasing in z_j for all $j \neq i$, then the method of vector Lyapunov functions can effectively be used with suitable conditions on V and F to obtain stability of the zero solution to (2.4); see [1-4].

In many situations in science and engineering, $f(t, V) = -AV$, where the matrix A is assumed to have positive diagonal and nonpositive off-diagonal elements. It is then well known that the comparison system $u' = -Au$ is stable iff A is a Metzler matrix. This method fails if A has positive off-diagonal elements. However, Theorem 2.1 is applicable in this situation. Since all $V_i \geq 0$, we can choose $a = 0$ and consequently we can get the desired estimates $V(t, z(t)) \leq r(t)$ for $t \geq t_0$, where r now is the quasi-maximal solution of the system

$$r'_i = f_i(t, r_i, [r]_{p_i}, \{0\}_{q_i}) \quad , \quad r(t_0) = V(t_0, z_0).$$

It is then a simple matter to obtain stability of the large scale system from the stability of the quasi-solutions.

Naturally, the comparison Theorem 2.1 can be used in place of the usual ones [5] in general situations to obtain qualitative behavior of a given system (2.4) under weaker hypotheses on the comparison system, that is, without requiring the quasi-monotone nondecreasing property of the comparison system. We shall be content in giving the following simple examples to demonstrate the usefulness of quasi-solutions.

Example 1. Consider $u' = Au$ where $a_{ii} < 0$, $a_{ij} \leq 0$ for $j < i$ and $a_{ij} \geq 0$ for $j > i$. The quasi-solutions (w.r. to $a = 0$) are determined by $v' = \tilde{A}v$, where $\tilde{a}_{ii} = a_{ii}$, $\tilde{a}_{ij} = 0$ for $j < i$ and $\tilde{a}_{ij} = a_{ij}$ for $j > i$. $\tilde{A}$ is stable since it has the eigenvalues $a_{11}, \dots, a_{nn} < 0$.

Example 2. Consider the system $x'_1 = -x_1 + x_2^{1/2}$, $x'_2 = 1 - x_1^2 - x_2$ with initial condition $x_1(0) = 2$, $x_2(0) = 1$. For $a = (1, 0)$ we get the quasi-solution $v(t) = (2e^{-t/2}, e^{-t})$. Thus, Theorem 2.1 implies, in particular, that $v_1(t_0) = 1$ for some $t_0 \leq 2 \log 2$. Obviously, $v(t) = (2e^{-t}, 0)$ and $w(t) = (2, 1)$ are coupled upper and lower quasi-solutions. Therefore $2e^{-t} \leq x_1(t) \leq 2e^{-t/2}$ and $x_2(t) \leq e^{-t}$ in $[0, \log 2]$.

3. Quasi-solutions and reaction-diffusion equations.

Let Ω be a bounded domain in $\mathbb{R}^N$ and $H = (0, T) \times \Omega$. We consider the system of nonlinear reaction-diffusion equations of the simple form

$$(3.1) \quad u_t = B\Delta u + f(t, u) \quad \text{in } (0, T) \times \Omega$$

with

$$(3.2) \quad u(0, x) = u_0(x) \text{ on } \bar{\Omega} \text{ and } \frac{\partial u}{\partial \tau} = 0 \text{ on } (0, T) \times \partial\Omega.$$

Here Δ denotes the Laplacean for $\mathbb{R}^N$, $u, f, u_0 \in \mathbb{R}^n$, B is a diagonal matrix and τ is the outer normal.

We shall assume that the functions in question have outer normal derivatives on $(0, T) \times \partial\Omega$, and if $u \in C(\bar{H}; \mathbb{R}^n)$ has partial derivatives u_t, u_x, u_{xx} which are continuous in H , we shall say that $u \in \mathcal{C}$.

Equations of the type (3.1) have been important in the modelling and study of population dynamics, chemical reactions and several models of nerve conduction; see e.g. [6] for various references. We have intentionally restricted ourselves to the simple equations (3.1), but our results extend to more general systems of the type

$$u_t^i = F_i(t, x, u, u_x^i, u_{xx}^i) \quad 1 \leq i \leq n,$$

following the technics given in [6].

A comparison theorem corresponding to Theorem 2.1 is the following

Theorem 3.1. Assume that

- (i) $v \in \mathcal{C}$, $v(0, x) \leq u_0(x)$ on $\bar{\Omega}$, $\frac{\partial v}{\partial \tau} \leq 0$ on $[0, T] \times \partial\Omega$, and $v(t, x) \geq a(t)$ for some $a \in C([0, T]; \mathbb{R}^n)$
- (ii) $v_t \leq B\Delta v + f(t, v)$ on H , $f \in C(Q_a; \mathbb{R}^n)$ and f has the mqmp.
- (iii) The maximal quasi-solution r (w.r.to a) of $w' = f(t, w)$, $w(0) = w_0 \geq v(0, x)$ on $\bar{\Omega}$ exists on $[0, T]$.
- (iv) There exists $z \in \mathcal{C}$ such that $z > 0$ on $\bar{H}$, $\frac{\partial z}{\partial \tau} \geq \gamma > 0$ on $(0, T] \times \partial\Omega$, and

$$\varepsilon z_t^i > \varepsilon b^i \Delta z^i + f_i(t, r_i + \varepsilon z_i, [r + \varepsilon z]_{p_i, \{a\} q_i}) - f_i(t, r_i, [r]_{p_i, \{a\} q_i})$$

on H , for all $\varepsilon > 0$ sufficiently small.

Then $v(t, x) \leq r(t)$ on $\bar{H}$.

Proof. Define $w(t, x) = r(t) + \varepsilon z(t, x)$ on $\bar{H}$. Then we have, using (iii) and (iv),

$$\begin{aligned} w_t^i &= r_i^i + \epsilon z_t^i > \epsilon b^i \Delta z^i + f_i(t, w^i, [w]_{p_i}, \{a\}_{q_i}) \\ &= b^i \Delta w^i + f_i(t, w^i, [w]_{p_i}, \{a\}_{q_i}) \text{ on } H. \end{aligned}$$

Also, $v(0, x) < w(0, x)$ on $\bar{\Omega}$, $\frac{\partial v}{\partial \tau}(t, x) < \frac{\partial w}{\partial \tau}(t, x)$ on $(0, T] \times \partial \Omega$ and

$$v_t^i \leq b^i \Delta v^i + f_i(t, v) \leq b^i \Delta v^i + f_i(t, [v]_{p_i}, \{a\}_{q_i}) \text{ on } H.$$

By means of standard arguments, see e.g. [5], [6], it follows that $v(t, x) < w(t, x)$ on $\bar{H}$, and letting $\epsilon \rightarrow 0$ yields the desired result.

Similarly one can formulate and prove the results corresponding to those given in section 2 with appropriate modifications. We shall merely state the following result.

Theorem 3.2. Assume that

- (i) u is any solution of (3.1), (3.2) on $\bar{H}$.
- (ii) v, w are coupled upper and lower quasi-solutions of $y' = f(t, y)$ such that $v(0) \leq u_0(x) \leq w(0)$ on $\bar{\Omega}$.
- (iii) There exists $z \in \mathcal{C}$ such that $z > 0$ on $\bar{H}$, $\frac{\partial z}{\partial \tau} \geq \gamma > 0$ on $(0, T] \times \partial \Omega$ and for all $\epsilon > 0$ sufficiently small

$$\epsilon z_t^i > \epsilon b^i \Delta z^i + f_i(t, w_i + \epsilon z^i, [w + \epsilon z]_{p_i}, \{v - \epsilon z\}_{q_i}) - f_i(t, w_i, [w]_{p_i}, \{v\}_{q_i})$$

and

$$\epsilon z_t^i > \epsilon b^i \Delta z^i + f_i(t, v_i, [v]_{p_i}, \{w\}_{q_i}) - f_i(t, v_i - \epsilon z^i, [v - \epsilon z]_{p_i}, \{w + \epsilon z\}_{q_i}).$$

Then $v(t) \leq u(t, x) \leq w(t)$ on $\bar{H}$.

The last conditions in the above theorems are satisfied if f is uniformly Lipschitz and $\partial \Omega$ is 'regular', i.e. there exists $h \in \mathcal{C}$ such that $\frac{\partial h}{\partial \tau}(x) \geq 1$ on $\partial \Omega$ and h_x, h_{xx} are bounded; see e.g. [6] for details.

4. Quasi-solutions of boundary value problems.

Let us finally consider boundary value problems and restrict ourselves to

$$(4.1) \quad -u_i'' = f_i(t, u, u_i') \text{ for } i = 1, \dots, n, \quad u(0) = c, \quad u(1) = d$$

on $J = [0,1]$, where $f \in C(J \times \mathbb{R}^n \times \mathbb{R}; \mathbb{R}^n)$. Splitting u as in section 2 we have

$$(4.2) \quad -u_i'' = f_i(t, u_i, [u]_{p_i}, \{u\}_{q_i}, u_i') , u(0) = c , u(1) = d.$$

Given $a \in C(J; \mathbb{R}^n)$, a function $r \in C^2(J; \mathbb{R}^n)$ is said to be a quasi-solution (w.r.to a) of (4.2) if for each i

$$-r_i'' = f_i(t, r_i, [r]_{p_i}, \{a\}_{q_i}, r_i') , r(0) = c, r(1) = d.$$

We then have the following comparison theorem.

Theorem 4.1. Let r be a quasi-solution (w.r. to a) of (4.2) and assume that

- (i) $v \in C^2(J, \mathbb{Q}_a)$, $-v_i'' \leq f_i(t, v, v_i')$, $v(0) \leq c$ and $v(1) \leq d$.
- (ii) There exists $z \in C^2(J, \mathbb{R}^n)$ such that $z > 0$ and for all $\lambda > 0$, $t \in (0,1)$
 $-\lambda z_i'' > f_i(t, r_i + \lambda z_i, [r + \lambda z]_{p_i}, \{a\}_{q_i}, r_i' + \lambda z_i') - f_i(t, r_i, [r]_{p_i}, \{a\}_{q_i}, r_i')$
- (iii) $f \in C(\mathbb{Q}_a \times \mathbb{R}; \mathbb{R}^n)$ and f has the mcmp w.r. to u .

Then $v(t) \leq r(t)$ on J .

Proof. If the conclusion is not true then there exist $\lambda > 0$, an index j and $t_0 \in (0,1)$ such that $v_j(t) \leq r_j(t) + \lambda z_j(t)$ in J for all i and $v_j(t_0) = r_j(t_0) + \lambda z_j(t_0)$. Hence we have $v_j'(t_0) = r_j'(t_0) + \lambda z_j'(t_0)$ and $v_j''(t_0) \leq r_j''(t_0) + \lambda z_j''(t_0)$. It therefore follows that at $t = t_0$

$$\begin{aligned} 0 \leq r_j'' + \lambda z_j'' - v_j'' &< -f_j(t_0, r_j + \lambda z_j, [r + \lambda z]_{p_j}, \{a\}_{q_j}, r_j' + \lambda z_j') + f_j(t_0, v, v_j') \\ &\leq -f_j(\dots) + f_j(t_0, v_j, [r + \lambda z]_{p_j}, \{a\}_{q_j}, v_j') = 0. \end{aligned}$$

This contradiction completes the proof.

Based on this result and those of section 2 it is easy to formulate and prove other corresponding results, the formulation of which is omitted here.

References:

- [1] Arakai, M.: Stability of large scale nonlinear systems. IEEE Trans. Auto. Control 23 (1978), 129-142.
- [2] Sandell, N.R.; Varaiya, P.; Athans, M. and Safanov, M.G.: Survey of decentralized control methods for large scale systems. IEEE Trans. Auto. Control. 23 (1978), 108-128.

- [3] Stijak, D.D.: Large scale dynamical systems. North Holland, New York 1978
- [4] Michel, A.N. and Miller, R.K.: Qualitative analysis of large scale dynamical systems. Academic Press, New York 1977.
- [5] Lakshmikantham, V. and Leela, S.: Differential and integral inequalities Vol. I, Academic Press 1969.
- [6] -, Comparison results for reaction-diffusion equations in a Banach space. Univ. of Texas at Arlington, Techn. Rep. No. 94 (1978).
- [7] Chandra, J.: Some comparison theorems for reaction diffusion equations. Proc. Int. Conf. "Recent Trends in Diff. Eqs.", Trieste 1978. Academic Press (to appear).
- [8] Gavalas, G.R.: Nonlinear differential equations of chemically reacting systems. Springer, New York 1968.
- [9] Bernfeld, S.R. and Lakshmikantham, V.: Introduction to nonlinear boundary value problems. Academic Press, New York 1974.
- [10] Deimling, K. and Lakshmikantham, V.: Existence and comparison theorems for differential equations in Banach spaces. Nonlinear Analysis TMA (to appear).
- [11] -, On existence of extremal solutions of differential equations in Banach spaces. Nonlinear Analysis TMA (to appear).