**GENERALIZED STABILITY OF MOTION
AND VECTOR LYAPUNOV FUNCTIONS**

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1. *Introduction.*

The direct theory for stability of motion in terms of vector Lyapunov functions and the general comparison method is well-developed [4, 5, 6, 7] and effectively applied for large scale dynamical systems [1, 3]. However, the problem of constructing vector Lyapunov functions and developing appropriate perturbation theory has seen very little progress. Perhaps one of the reasons for this situation is that the stability definitions by means of the standard norm are not flexible enough for such a venture.

It is known [2, 5, 8] that a generalized space where the norm is vector valued offers a more flexible mechanism than the one which is normally used and is advantageous in certain situations. In this paper we work with generalized spaces, thus allowing us to introduce the concepts of generalized stability of motion and also boundedness. To avoid monotony we will only give a sampling of definitions and theorems which can be obtained using generalized norms.

In Section 3 we will state and prove three theorems concerning different properties of certain subspaces of R^n . One should note that

these properties can be exhibited using generalized norms but could easily be unobservable if an ordinary norm were employed in the investigation. Finally, we will state a theorem combining the three results, thus emphasizing the advantages of using the generalized norm.

2. Preliminaries

In the following definitions and theorems we will use the vectorial inequality with the understanding that the same inequalities hold between corresponding components.

Definition 2.1. A generalized norm from R^n to R^k is a mapping $||\cdot||_G : R^n \rightarrow R^k$ denoted by $||x||_G = (\alpha_1(x), \dots, \alpha_k(x))$ such that

- (a) $||x||_G \geq 0$, (that is $\alpha_i(x) \geq 0$ for $i = 1, \dots, k$);
- (b) $||x||_G = 0$ if and only if $x = 0$, (that is $\alpha_i(x) = 0$ for $i = 1, \dots, k$ if and only if $x = 0$);
- (c) $||\lambda x||_G = |\lambda| ||x||_G$, (that is $\alpha_i(\lambda x) = |\lambda| \alpha_i(x)$ for $i = 1, \dots, k$);
- (d) $||x + y||_G \leq ||x||_G + ||y||_G$, (that is $\alpha_i(x + y) \leq \alpha_i(x) + \alpha_i(y)$).

It is important to note that all generalized norms (which includes all norms) generate the same topology on R^n . In particular, it is easy to show that the α_i 's are ordinary norms when restricted to appropriate subspaces K_i of R^n . Furthermore, these K_i 's span R^n .

Consider the generalized norm space $E = (R^n, ||\cdot||_G)$ and the following differential systems:

$$(2.1) \quad x' = f(t, x), \quad x(t_0) = x_0$$

$$(2.2) \quad u' = g(t, u), \quad u(t_0) = u_0 \geq 0$$

where $f \in C[R_+ \times S(p), R^n]$ and $g \in C[R_+ \times R_+^m, R^m]$. Here $R_+ = [0, \infty)$, $C[R_+ \times S(p), R^n]$ is the class of all continuous functions from $R_+ \times S(p)$ to R^n and $S(p) = \{x \in E \mid \|x\|_G < p\}$ with $p \in R_+^k$ and $0 < p_i \leq \infty$ for $i = 1, \dots, k$.

Definition 2.2. The trivial solution $x = 0$ of (2.1) is equistable if, for each $\varepsilon \in R_+^k$, $\varepsilon > 0$, and $t_0 \in R_+$, there exists a function $0 < \delta = \delta(t_0, \varepsilon)$ in R_+^k that is continuous in t_0 for each ε such that if $\|x_0\|_G \leq \delta$, then $\|x(t, t_0, x_0)\|_G < \varepsilon$ for $t \geq t_0$.

Definition 2.3. The trivial solution $x = 0$ of (2.1) is equiasymptotically stable if $x = 0$ is equistable and for each $\varepsilon \in R_+^k$, $\varepsilon > 0$ and $t_0 \in R_+$, there exists $\delta_0 = \delta_0(t_0)$ in R_+^k , $\delta_0 > 0$ and $T = T(t_0, \varepsilon)$ such that if $t \geq t_0 + T$ and $\|x_0\|_G < \delta_0$ then $\|x(t, t_0, x_0)\|_G < \varepsilon$.

Definition 2.4. If all solutions of (2.1) exist globally, then the system (2.1) is equibounded if, for each $\gamma \in R_+^k$ and $t_0 \in R_+$, there exists a function $0 < \beta = \beta(t_0, \gamma)$ in R_+^k that is continuous in t_0 for each γ such that if $\|x_0\|_G \leq \gamma$, then $\|x(t, t_0, x_0)\|_G < \beta$ for $t \geq t_0$.

Definition 2.5. Let $(\alpha_1(x), \dots, \alpha_p(x))$ be a generalized norm when restricted to the subspace K of R^n . (That is, a projection of

$\|x\|_G$ yields a generalized norm on the subspace K). Then the trivial solution of (2.1) is partially equistable with respect to the subspace K if, for each $\varepsilon \in R_+^k$, (or $\varepsilon \in R_+^r$), $\varepsilon > 0$ and $t_0 \in R_+$, there exists a function $0 < \delta = \delta(t_0, \varepsilon)$ in R_+^k that is continuous in t_0 for each ε such that if $\|x_0\|_G \leq \delta$, then $(\|x(t, t_0, x_0)\|_G)_i < \varepsilon_i$ for $t \geq t_0$, $i = 1, \dots, r$. (That is $\alpha_i(x(t, t_0, x_0)) < \varepsilon_i$ for $t \geq t_0$, $i = 1, \dots, r$).

We can formulate similar definitions for partial equiboundedness of (2.1) and partial equi-asymptotic stability of the solution $x = 0$ of (2.1). Also we can make similar definitions for the system (2.2) by using the generalized norm $\|u\|_G^\wedge = (|u_1|, |u_2|, \dots, |u_m|)$. Since $u \in R_+^m$, then $\|u\|_G^\wedge = (u_1, u_2, \dots, u_m)$. Finally, we can make similar definitions for equistability of $u = 0$ in the first s coordinates ($s \leq m$), equi-asymptotic stability of $u = 0$ in the first s coordinates ($s \leq m$), and for equiboundedness of (2.2) in the first s coordinates ($s \leq m$) by restricting $\varepsilon, \delta, \gamma$, and β to being in R_+^s and only comparing in the first s coordinates (or only comparing in the first s coordinates and choosing $\varepsilon, \delta, \gamma$, and β as before).

We will also need a generalization of a function of class K (see [5]).

Definition 2.6. A function $\phi \in C[R_+^r, R_+^s]$ is said to belong to class $\hat{K}$ if $\phi(v) = 0$ if and only if $v = 0$ and $v \leq w$ implies $\phi(v) \leq \phi(w)$.

Finally, when considering the system (2.1), if $V \in C[R_+ \times S(p), R_+^m]$, then throughout this paper we define $D^+V(t, x)$ for all $(t, x) \in R_+ \times S(p)$

by

$$D^+V(t,x) = \limsup_{h \rightarrow 0^+} \frac{1}{h} [V(t+h, x+h f(t,x)) - V(t,x)].$$

3. Main Results

The statements of the first three theorems considered below appear to be quite complicated. However, this is primarily due to the fact that we are really working with subspaces. For clarity a corollary is given after each theorem.

Theorem 3.1. Assume that

- (1) $f \in C[R_+ \times S(p), R^n]$, $f(t,0) \equiv 0$;
- (2) $g \in C[R_+ \times R_+^m, R^m]$, $g(t,0) \equiv 0$, g is quasimonotone non-decreasing;
- (3) there exists $V \in C[R_+ \times S(p), R_+^m]$, $V(t,0) \equiv 0$, $V(t,x)$ is locally Lipschitzian in x ;
- (4) if $\|x\|_G = (\alpha_1(x), \dots, \alpha_n(x))$, then $(\alpha_1(x), \dots, \alpha_n(x))$ is a generalized norm when restricted to the subspace K of R^n ;
- (5) there exists $b \in \hat{K}$, $b : R_+^n \rightarrow R_+^s$ such that for

$$(t,x) \in R_+ \times S(p), b_i((\alpha_1(x), \dots, \alpha_n(x))) \leq V_i(t,x),$$

$$i = 1, \dots, s \leq m;$$
- (6) $D^+V(t,x) \leq g(t, V(t,x))$, $(t,x) \in R_+ \times S(p)$.

Then the equistability (or partial equistability) in the first s coordinates of the trivial solution of (2.2) implies the trivial solution of

(2.1) is partially equistable with respect to the subspace K .

Proof. We will do the case for equistability noting that an almost identical proof will work for the partial equistability case. Let $0 < \varepsilon < p$, $t_0 \in R_+$ be given. By hypothesis (5) there exists a function $b(v)$ of class $\hat{K}$ such that

$$b_i((\alpha_1(x), \dots, \alpha_r(x))) \leq V_i(t, x), \quad (t, x) \in R_+ \times S(p), \\ i = 1, \dots, s.$$

Let $\psi = \min\{b_i((0, \dots, 0, \varepsilon_j, 0, \dots, 0)) \mid 1 \leq i \leq s, 1 \leq j \leq r, \text{ and } b_i((0, \dots, 0, \varepsilon_j, 0, \dots, 0)) \neq 0\}$.

If η is the s -dimensional vector with all entries equal to ψ , then $\eta > 0$. Also, for each i , there exists $k(i)$ such that $\eta_{k(i)} \leq b_{k(i)}((0, \dots, 0, \varepsilon_i, 0, \dots, 0))$.

Now the equistability in the first s coordinates of $u = 0$ of (2.2) implies that given $\eta > 0$ and $t_0 \in R_+$, there exists a function $0 < \hat{\delta} = \hat{\delta}(t_0, \varepsilon)$ in R_+^m that is continuous in t_0 for each ε , such that

$$(u(t, t_0, u_0))_i < \eta_i, \quad t \geq t_0 \quad \text{if} \quad (u_0)_i \leq (\hat{\delta})_i, \quad i = 1, \dots, s.$$

Choose $u_0 = V(t_0, x_0)$. Since $V(t, x)$ is continuous and $V(t, 0) \equiv 0$, there exists a function $0 < \delta = \delta(t_0, \varepsilon)$ in R_+^k that is continuous in t_0 for each ε such that if $\|x_0\|_G \leq \delta$, then $V_i(t_0, x_0) \leq \hat{\delta}_i$, $i = 1, \dots, s$.

We claim that if $\|x_0\|_G \leq \delta$, then $\alpha_i(x(t, t_0, x_0)) < \varepsilon_i$ for $t \geq t_0$ and $i = 1, \dots, r$. Suppose this is not true. Then there exists a solution $x(t) = x(t, t_0, x_0)$ with $\|x_0\|_G \leq \delta$ and a $t_1 > t_0$ such that $\alpha_i(x(t)) < \varepsilon_i$ for $t \in [t_0, t_1)$ and $i = 1, \dots, r$, and there exists an h such that $\alpha_h(x(t_1)) = \varepsilon_h$ where $1 \leq h \leq r$. Now by hypothesis (5)

$$b_i((\alpha_1(x(t_1)), \dots, \alpha_r(x(t_1)))) \leq V_i(t_1, x(t_1)), \quad i = 1, \dots, s.$$

Since $\|x(t)\|_G < p$ for $t \in [t_0, t_1]$ and $u_0 = V(t_0, x_0)$, then using hypothesis (6) and applying Theorem 4.1.1 in [5], we obtain

$$V(t, x(t)) \leq r(t, t_0, u_0), \quad t \in [t_0, t_1],$$

where $r(t, t_0, u_0)$ is the maximal solution of (2.2). Thus, we have

$$V_i(t_1, x(t_1)) < \eta_i, \quad i = 1, \dots, s.$$

However, for the h such that $\alpha_h(x(t_1)) = \varepsilon_h$ recall that

$$\eta_{k(h)} \leq b_{k(h)}((0, \dots, 0, \varepsilon_h, 0, \dots, 0)).$$

This leads to the contradiction

$$\eta_{k(h)} \leq b_{k(h)}((0, \dots, 0, \varepsilon_h, 0, \dots, 0)) \leq V_{k(h)}(t, x(t)) < \eta_{k(h)},$$

proving that the trivial solution of (2.1) is partially equistable with respect to the subspace K .

Corollary 3.1. Assume that

$$(1) \quad f \in C[R_+ \times S(p), R^n], \quad f(t, 0) \equiv 0;$$

- (2) $g \in C[R_+ \times R_+^m, R^m]$, $g(t, 0) \equiv 0$, g is quasimonotone nondecreasing in u ;
- (3) there exists $V \in C[R_+ \times S(p), R_+^m]$, $V(t, 0) \equiv 0$, $V(t, x)$ is locally Lipschitzian in x ;
- (4) there exists $b \in \hat{K}$, $b : R_+^k \rightarrow R_+^m$ such that for $(t, x) \in R_+ \times S(p)$, $b(\|x\|_G) \leq V(t, x)$;
- (5) $D^+V(t, x) \leq g(t, V(t, x))$ for $(t, x) \in R_+ \times S(p)$.

Then the equistability of the trivial solution of (2.2) implies the trivial solution of (2.1) is equistable.

We shall next prove a typical result covering boundedness of solutions.

Theorem 3.2. Assume that

- (1) $f \in C[R_+ \times R^n, R^n]$ and all solutions of (2.1) exist globally;
- (2) $g \in C[R_+ \times R_+^m, R^m]$, g is quasimonotone nondecreasing in u ;
- (3) there exists $V \in C[R_+ \times R^n, R_+^m]$, $V(t, 0) \equiv 0$, $V(t, x)$ is locally Lipschitzian in x ;
- (4) if $\|x\|_G = (\alpha_1(x), \dots, \alpha_k(x))$, then $(\alpha_1(x), \dots, \alpha_p(x))$ is a generalized norm when restricted to the subspace K of R^n ;
- (5) there exists $b \in \hat{K}$, $b : R_+^r \rightarrow R_+^s$ such that for $(t, x) \in R_+ \times R^n$, $b_i((\alpha_1(x), \dots, \alpha_p(x))) \leq V_i(t, x)$, $i = 1, \dots, s \leq m$;
- (6) for each i such that $1 \leq i \leq r$ there exists $j(i)$ such that $\lim_{v_i \rightarrow \infty} b_{j(i)}(0, \dots, 0, v_i, 0, \dots, 0) = \infty$;
- (7) $D^+V(t, x) \leq g(t, V(t, x))$, $(t, x) \in R_+ \times R^n$.

Then the equiboundedness (or partial equiboundedness) in the first s coordinates of the system (2.2) implies the system (2.1) is partially equibounded with respect to the subspace K .

Proof. We will do the case for equiboundedness again noting that an almost identical proof will work for the partial equiboundedness case. Let $\gamma \in R_+^k$ and $t_0 \in R_+$ be given, and let $\|x_0\|_G \leq \gamma$. By hypothesis (3) there exists a number $\hat{\gamma} = \hat{\gamma}(t_0, \gamma)$ such that $V(t_0, x_0) \leq \hat{\gamma}$ whenever $\|x_0\|_G \leq \gamma$. Assume that the system (2.2) is equibounded in the first s coordinates. Then given $\hat{\gamma} \geq 0$ and $t_0 \in R_+$, there exists a positive vector $\beta = \beta(t_0, \gamma) = (\beta_1, \dots, \beta_s)$ that is continuous in t for each γ such that $u(t, t_0, u_0) < \beta$ for $t \geq t_0$ whenever $(u_0)_i \leq \hat{\gamma}_i$, $i = 1, \dots, s$ and $u(t, t_0, u_0)$ is a solution of (2.2).

Let $\eta_i = \sup \{\alpha_i \mid b_j((\alpha_1, \dots, \alpha_i, \dots, \alpha_r)) < \beta_j \text{ for all } j = 1, \dots, s\}$ for each $i = 1, \dots, r$. Note that since $b \in \hat{K}$ then $\eta_i = \sup \{\alpha_i \mid b_j((0, \dots, 0, \alpha_i, 0, \dots, 0)) < \beta_j \text{ for all } j = 1, \dots, s\} > 0$. Furthermore, hypothesis (6) yields $\eta_i < \infty$. Also for each i , there exists some $k(i)$ such that

$$b_{k(i)}((0, \dots, 0, \eta_i, 0, \dots, 0)) = \beta_{k(i)}$$

Let $\eta = (\eta_1, \dots, \eta_r)$ and $u_0 = V(t_0, x_0)$. Then hypothesis (7) and Theorem 4.1.1 in [5] give us that

$$V(t, x(t, t_0, x_0)) \leq r(t, t_0, u_0), \quad t \geq t_0$$

where $r(t, t_0, u_0)$ is the maximal solution of (2.2).

We claim that if $\|x_0\|_G \leq \gamma$, then $(\alpha_1(x), \dots, \alpha_r(x)) \leq \eta$.

Suppose this is not true. Then there exists a solution $x(t) = x(t, t_0, x_0)$ with $\|x_0\|_G \leq \gamma$ and a $t_1 > t_0$ such that $(\alpha_1(x(t)), \dots, \alpha_n(x(t))) < \eta$ for $t \in [t_0, t_1)$ and there exists an h such that $\alpha_h(x(t_1)) = \eta_h$. However, for this h , there exists some $k(h)$ such that

$$b_{k(h)}((0, \dots, 0, \eta_h, 0, \dots, 0)) = \beta_{k(h)}.$$

This leads to the contradiction

$$\begin{aligned} \beta_{k(h)} &= b_{k(h)}((0, \dots, 0, \eta_h, 0, \dots, 0)) \\ &\leq V_{k(h)}(t_1, x(t_1)) \leq r_{k(h)}(t_1, t_0, u_0) < \beta_{k(h)}. \end{aligned}$$

Hence, we have the system (2.1) is partially equibounded with respect to the subspace K .

Corollary 3.2. Assume that

- (1) $f \in C[R_+ \times R^n, R^n]$ and all solutions of (2.1) exist globally;
- (2) $g \in C[R_+ \times R^m, R^m]$, $g(t, 0) \equiv 0$, g is quasimonotone non-decreasing in u ;
- (3) there exists $V \in C[R_+ \times R^n, R^m]$, $V(t, 0) \equiv 0$, $V(t, x)$ is locally Lipschitzian in x ;
- (4) there exists $b \in \hat{K}$, $b : R_+^k \rightarrow R_+^m$ such that for $(t, x) \in R_+ \times S(p)$, $b(\|x\|_G) \leq V(t, x)$;
- (5) for each i such that $1 \leq i \leq k$ there exists $j(i)$ such that $\lim_{v_i \rightarrow \infty} b_{j(i)}((0, \dots, 0, v_i, 0, \dots, 0)) = \infty$;
- (6) $D^+V(t, x) \leq g(t, V(t, x))$, $(t, x) \in R_+ \times R^n$.

Then the equiboundedness of the system (2.2) implies the system (2.1) is equibounded.

The next result deals with asymptotic stability.

Theorem 3.3. Assume that

- (1) $f \in C[R_+ \times S(p), R^n]$, $f(t, 0) \equiv 0$;
- (2) $g \in C[R_+ \times R_+^m, R^m]$, $g(t, 0) \equiv 0$,
 g is quasimonotone nondecreasing in u ;
- (3) there exists $V \in C[R_+ \times S(p), R_+^m]$, $V(t, 0) \equiv 0$,
 $V(t, x)$ is locally Lipschitzian in x ;
- (4) if $\|x\|_G = (\alpha_1(x), \dots, \alpha_k(x))$, then $(\alpha_1(x), \dots, \alpha_k(x))$
 is a generalized norm when restricted to the subspace
 K of R^n ;
- (5) there exists $b \in \hat{K}$, $b : R_+^n \rightarrow R_+^s$ such that for
 $(t, x) \in R_+ \times S(p)$, $b_i((\alpha_1(x), \dots, \alpha_k(x))) \leq V_i(t, x)$,
 $i = 1, \dots, s \leq m$;
- (6) $D^+V(t, x) \leq g(t, V(t, x))$, $(t, x) \in R_+ \times S(p)$.

Then the equi-asymptotic stability (or partial equi-asymptotic stability) in the first s coordinates of the trivial solution of (2.2) implies the trivial solution of (2.1) is partially equi-asymptotically stable with respect to the subspace K .

Proof. Again we will do the case for equi-asymptotic stability noting the partial equi-asymptotic stability case has an almost identical proof. Suppose the trivial solution of (2.2) is equi-asymptotically stable in the first s coordinates. Then by Theorem 3.1

we know the trivial solution of (2.1) is partially equistable with respect to K . Let $0 < \varepsilon < p$, $t_0 \in R_+$ be given. By hypothesis (5), there exists a function $b(v)$ of class $\hat{K}$ such that

$$b_i((\alpha(x), \dots, \alpha_r(x))) \leq V_i(t, x), \quad (t, x) \in R_+ \times S(p), \quad i=1, \dots, s.$$

Let $\psi = \min \{b_i((0, \dots, 0, \varepsilon_j, 0, \dots, 0)) \mid 1 \leq i \leq s, 1 \leq j \leq r, \text{ and } b_i((0, \dots, 0, \varepsilon_j, 0, \dots, 0)) \neq 0\}$.

If η is the s -dimensional vector with all entries equal to ψ , then $\eta > 0$. Also, for each i , there exists $k(i)$ such that $\eta_{k(i)} \leq b_{k(i)}((0, \dots, 0, \varepsilon_i, 0, \dots, 0))$.

Now the trivial solution of (2.2) is equi-asymptotically stable in the first s coordinates, so it follows that given $\eta > 0$, $t_0 \in R_+$, there exists a positive number $T = T(t_0, \varepsilon)$ and a function $0 < \hat{\delta} = \hat{\delta}(t_0)$ in R_+^m such that

$$(u(t, t_0, u_0))_i < \eta_i, \quad t \geq t_0 + T, \quad i = 1, \dots, s$$

if $(u_0)_i \leq \hat{\delta}_i, \quad i = 1, \dots, s.$

Choose $u_0 = V(t_0, x_0)$. Since $V(t, x)$ is continuous and $V(t, 0) \equiv 0$, there exists a function $0 < \delta = \delta(t_0)$ such that if $\|x_0\|_G \leq \delta$, then $V_i(t_0, x_0) \leq \delta_i, \quad i = 1, \dots, s.$

We claim that if $\|x_0\|_G \leq \delta$, then $\alpha_i(x(t, t_0, x_0)) < \varepsilon_i$ for $t \geq t_0 + T$ and $i = 1, \dots, r$. Suppose this is not true. Then there exists a solution $x(t, t_0, x_0)$ with $\|x_0\|_G \leq \delta$ and there exists a sequence $\{t_n\}_{n=1}^\infty$ where $t_n \geq t_0 + T$ and $\lim_{n \rightarrow \infty} t_n = \infty$ such that

$\alpha_h(x(t_n, t_0, x_0)) \geq \varepsilon_h$ for some h where $1 \leq h \leq r$. Now proceeding as in the proof of Theorem 3.1 we are lead to the contradiction.

$$\eta_{k(h)} \leq b_{k(h)}((0, \dots, 0, \varepsilon_h, 0, \dots, 0)) \leq v_{k(h)}(t_n, x(t_n, t_0, x_0)) < \eta_{k(h)} \\ \text{for all } t_n.$$

Thus, the trivial solution is partially equi-asymptotically stable with respect to the subspace K .

Corollary 3.3. Assume that

- (1) $f \in C[R_+ \times S(p), R^n]$, $f(t, 0) \equiv 0$;
- (2) $g \in C[R_+ \times R_+^m, R^m]$, $g(t, 0) \equiv 0$, g is quasimonotone nondecreasing in u ;
- (3) there exists $V \in C[R_+ \times S(p), R_+^m]$, $V(t, 0) \equiv 0$,
 $V(t, x)$ is locally Lipschitzian in x ;
- (4) there exists $b \in \hat{K}$, $b : R_+^k \rightarrow R_+^m$ such that for
 $(t, x) \in R_+ \times S(p)$, $b(|x|_G) \leq V(t, x)$;
- (5) $D^+V(t, x) \leq g(t, V(t, x))$, $(t, x) \in R_+ \times S(p)$.

Then the equi-asymptotic stability of the trivial solution of (2.2) implies the trivial solution of (2.1) is equi-asymptotically stable.

Combining the ideas of the foregoing three theorems, we can now state a result which demonstrates the greater sensitivity of the generalized norm. We merely state the result since the proof is a direct consequence of Theorems 3.1, 3.2, and 3.3.

Theorem 3.4. Assume that

- (1) $f \in C[R_+ \times R^n, R^n]$, $f(t, 0) \equiv 0$, and all solutions of (2.1) exist globally;
- (2) $g \in C[R_+ \times R^m, R^m]$, $g(t, 0) \equiv 0$, g is quasimonotone nondecreasing in u ;
- (3) there exists $V \in C[R_+ \times R^n, R^m]$, $V(t, 0) \equiv 0$, $V(t, x)$ is locally Lipschitzian in x ;
- (4) if $\|x\|_G = (\alpha_1(x), \dots, \alpha_{r_1}(x), \dots, \alpha_{r_1+r_2}(x), \dots, \alpha_{r_1+r_2+r_3}(x), \dots, \alpha_k(x))$, then $(\alpha_1(x), \dots, \alpha_{r_1}(x))$, $(\alpha_{r_1+1}(x), \dots, \alpha_{r_1+r_2}(x))$, and $(\alpha_{r_1+r_2+1}(x), \dots, \alpha_{r_1+r_2+r_3}(x))$ are generalized norms when restricted respectively to subspaces K_1 , K_2 , and K_3 of R^n ;
- (5) there exists b^j , $j = 1, 2, 3$, such that $b^j : R_+^{r_j} \rightarrow R_+^{s_j}$ and $b^j \in \hat{K}$ for $j = 1, 2, 3$; and for $(t, x) \in R_+ \times R^n$

$$b_i^1((\alpha_1(x), \dots, \alpha_{r_1}(x))) \leq V_i(t, x), \quad i = 1, \dots, s_1$$

$$b_i^2((\alpha_{r_1+1}(x), \dots, \alpha_{r_1+r_2}(x))) \leq V_{s_1+i}(t, x), \quad i = 1, \dots, s_2$$

$$b_i^3((\alpha_{r_1+r_2+1}(x), \dots, \alpha_{r_1+r_2+r_3}(x))) \leq V_{s_1+s_2+i}(t, x),$$

$$i = 1, \dots, s_3$$
- (6) for each i such that $r_1 + 1 \leq i \leq r_1 + r_2$ there exists $j(i)$ such that $\lim_{v_i \rightarrow \infty} b_{j(i)}^2((0, \dots, 0, v_i, 0, \dots, 0)) = \infty$;
- (7) $D^+V(t, x) \leq g(t, V(t, x))$, $(t, x) \in R_+ \times R^n$.

Then, if the trivial solution of (2.2) is equistable in the first s_1 coordinates, the system (2.2) is equibounded in the coordinates $s_1 + 1, \dots, s_1 + s_2$, and the trivial solution of (2.2) is equiasymptotically stable in the coordinates $s_1 + s_2 + 1, \dots, s_1 + s_2 + s_3$, then the trivial solution of (2.1) is partially equistable with respect to the subspace K_1 , the system (2.1) is partially equibounded with respect to the subspace K_2 , and partially equiasymptotically stable with respect to the subspace K_3 .

4. Example

Consider the system

$$\begin{aligned}
 x'_1 &= x_1 e^{-t} + x_2 \sin t - x_1 x_4^2 \\
 x'_2 &= x_1 \sin t + x_2 e^{-t} - x_2 x_4^2 \\
 (4.1) \quad x'_3 &= e^{-t} - x_2^2 x_3 \\
 x'_4 &= (e^{-t} - 2) x_4 - x_1^2 x_4 \\
 x'_5 &= -3x_5 - x_2^2 x_5,
 \end{aligned}$$

for $t \geq 0$.

Suppose we choose the vector Lyapunov function

$$V(t, x) = \begin{bmatrix} V_1(t, x) \\ V_2(t, x) \\ V_3(t, x) \\ V_4(t, x) \end{bmatrix}$$

where $\|x\|_G^2$ means square component-wise,

$$x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}, \text{ and } ||x||_G = \begin{bmatrix} |x_1 + x_2| \\ |x_1 - x_2| \\ |x_3| \\ \sqrt{x_4^2 + x_5^2} \end{bmatrix}.$$

$$\text{Let } b^1 \left(\begin{bmatrix} |x_1 + x_2| \\ |x_1 - x_2| \end{bmatrix} \right) = \begin{bmatrix} (x_1 + x_2)^2 \\ (x_1 - x_2)^2 \end{bmatrix}, \quad b^2(|x_3|) = x_3^2,$$

$$\text{and } b^3(\sqrt{x_4^2 + x_5^2}) = x_4^2 + x_5^2. \text{ Then}$$

$$V_1 \geq b_1^1, \quad V_2 \geq b_2^1, \quad V_3 \geq b_3^2, \quad \text{and } V_4 \geq b_4^3.$$

Now,

$$\begin{aligned} D_+ V_1(t, x) &= 2(x_1 + x_2)(x_1' + x_2') \\ &= 2(x_1 + x_2)^2(e^{-t} + \sin t) - 2x_4^2(x_1 + x_2)^2 \\ &\leq 2V_1(t, x)(e^{-t} + \sin t). \end{aligned}$$

$$\text{Similarly, } D_+ V_2(t, x) \leq 2V_2(t, x)(e^{-t} - \sin t),$$

$$D_+ V_3(t, x) \leq 2\sqrt{V_3} e^{-t},$$

$$D_+ V_4(t, x) \leq -2V_4.$$

Consider the comparison equation

$$u' = g(t, u) \quad \text{where } u = \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} \quad \text{and } t \geq 0,$$

given by

$$\begin{aligned}
 (4.2) \quad u_1^1 &= 2u_1(e^{-t} + \sin t) \\
 u_2^1 &= 2u_2(e^{-t} - \sin t) \\
 u_3^1 &= 2\sqrt{u_3} e^{-t} \\
 u_4^1 &= -2u_4.
 \end{aligned}$$

Now the solution $u \equiv 0$ of (4.2) is equistable in u_1 and u_2 , equibounded in u_3 , and equi-asymptotically stable in u_4 .

Applying Theorem 3.4, the solution $x \equiv 0$ of (4.1) is equistable in x_1 and x_2 , equibounded in x_3 , and equi-asymptotically stable in x_4 and x_5 .

REFERENCES

1. Bailey, F. N., The Application of Lyapunov's Second Method to Interconnected Systems, *J. SIAM Control*, 3(1966), 443-462.
2. Bernfeld, S., and Lakshmikantham, V., *An Introduction to Nonlinear Boundary Value Problems*, Academic Press: New York, 1974.
3. Grujic, L.T., and Siljak, D.D., Asymptotic Stability and Instability of Large Scale Systems, *IEEE Trans. Automatic Control* (1973).
4. Lakshmikantham, V., Several Lyapunov Functions, *Proc. Int. Conf. on Nonlinear Oscillations*, Kiev, U.S.S.R., (1970).
5. Lakshmikantham, V., and Leela, S., *Differential and Integral Inequalities*, Vol. 1, Academic Press: New York, 1969.
6. Lakshmikantham, V., and Leela, S., Global Results and Stability of Motion, *Proc. Camb. Phi. Soc.* 70(1971), 95-102.
7. Matrasov, V. M., Vector Lyapunov Functions in the Analysis of Nonlinear Interconnected Systems, *Sympos. Math Roma*, 6, *Mecanica Nonlineare Stabilita* 1970, (1971), 209-262.
8. Mitchell, Roger W., and Moore, Marion E., Vector Lyapunov Functions and Perturbations of Nonlinear Systems, University of Texas at Arlington, Arlington, Texas, Technical Report #23.