THE METHOD OF QUASILINEARIZATION AND POSITIVITY
OF SOLUTIONS IN ABSTRACT CONES

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1. Introduction.

The method of quasilinearization was first introduced by Bellman (Ref. 1) and was developed further by Kalaba (Ref. 2). More detailed information concerning the method can be found in (Ref. 3). This method has been very useful in providing an analytic approach to obtain approximate solutions of non-linear equations, or more specifically, to derive upper and lower bounds for the solutions.

In the study of qualitative properties of solutions of differential equations, it is well known (Ref. 4) that the general comparison principle has been a very powerful and versatile technique. Theoretically this technique reduces the problem of studying a system of differential equations to a relatively simple comparison differential equation. For deriving properties of solutions of the general comparison equation, other methods are necessary. The method of quasilinearization is a useful technique in this direction.

We wish to extend this fruitful idea to differential equations in a Banach space through abstract cones as to unify the theory and to offer more flexibility in applications. For this purpose, it becomes

necessary to consider the existence of maximal solutions and an abstract comparison principle. Since comparison systems represent abstract models for competitive processes that exist in several areas and since the most striking property of such systems is that the solutions must belong to a positive cone, (Refs. 5, 6), we also investigate the positivity of solutions. The method of quasilinearization is then developed in a systematic way. The flexibility inherent in the abstract extension is demonstrated in detail.

2. Preliminaries.

Throughout this paper E will denote a real Banach space with norm $||\cdot||$. A nonvoid closed subset K of E is called a positive cone if (i) $x, y \in K$ and $\alpha, \beta \geq 0$ implies $\alpha x + \beta y \in K$, and (ii) $x, -x \in K$ implies $x \equiv \theta$, where θ is the zero element of E . Furthermore, we will assume K° (the interior of K) is nonvoid. Two partial orderings are induced by K : $x \leq y$ if $y - x \in K$, and $x < y$ if $y - x \in K^\circ$. The following lemma (Ref. 7) incorporates some basic properties of these partial orderings.

Lemma 2.1: Let K be a cone in E . Then

- 1) $x \leq y, y < z$ implies $x < z$;
- 2) $x \leq y, z < w$ implies $x + z < y + w$;
- 3) $x_n \leq y_n$ for each $n, x_n \rightarrow x, y_n \rightarrow y$ implies $x \leq y$.

A cone K is said to be regular if each monotonic bounded sequence has a limit. The symbol K^* will denote $\{c : c \text{ is a con-}$

tinuous linear functional on E with $cx > 0$ for all $x \in K^0$. A function $f : E \rightarrow E$ is said to be quasimonotone nondecreasing with respect to K if whenever $x \leq y$ and $c \in K^*$ with $cx = cy$, then $cf(x) \leq cf(y)$, (Ref. 8).

Let $f \in C[[t_0, t_0 + \alpha] \times E, E]$. If at a point $x \in E$,

$$f(t, x + h) = f(t, x) + L(t, x, h) + \|h\| \eta(t, x, h), \quad h \in E$$

where $L(t, x, \cdot)$ is a linear operator and $\lim_{\|h\| \rightarrow 0} \|\eta(t, x, h)\| = 0$. Then $L(t, x, h)$ is called the Fréchet differential of the function f at the point x with increment h and $\eta(t, x, h)$ is called the remainder of the differential. The operator $L(t, x, \cdot) : E \rightarrow E$ is called the Fréchet derivative of f at x and is denoted by $f'_x(t, x)$, see (Ref. 9).

The following theorem is a consequence of a result concerning differential inequalities proved in (Ref. 8). We have stated it in a familiar version.

Theorem 2.1: Let $f \in C[[t_0, t_0 + \alpha] \times E, E]$, $f(t, x)$ is quasimonotone nondecreasing in x for each $t \in [t_0, t_0 + \alpha]$, and K is a cone in E with $K^0 \neq \emptyset$. Let $x, y \in C[[t_0, t_0 + \alpha], E]$. Suppose that $x'(t) \leq f(t, x(t))$ and $y'(t) \geq f(t, y(t))$ for $t \in [t_0, t_0 + \alpha]$, one of the inequalities being strict. Then $x(t_0) < y(t_0)$ implies $x(t) < y(t)$, $t \in [t_0, t_0 + \alpha]$.

3. Positivity of Solutions.

Let $D \subset E$ be an open set and let $f \in C[[t_0, t_0 + \alpha] \times D, E]$.

We consider the initial value problem

$$x' = f(t, x), \quad x(t_0) = x_0. \quad (1)$$

Let $S[x_0, b]$ denote $\{x \in E : \|x - x_0\| \leq b\}$. Let us assume that (H_0) there exist $M > 0, b > 0$ such that $S[x_0, b] \subset D$,

$$\|f(t, x)\| \leq M \text{ on } [t_0, t_0 + a] \times S[x_0, b], \text{ and } Ma \leq b.$$

We begin by proving a useful convergence result.

Theorem 3.1: Assume that (H_0) holds. Suppose further that

(1) $f_n \in C[[t_0, t_0 + a] \times D, E]$ for $n \geq 1$, and the sequence $\{f_n\}$ converges uniformly to f on $[t_0, t_0 + a] \times S[x_0, b]$;

(2) for each $n \geq 1$, $x_n(t)$ is a solution of $x' = f_n(t, x)$, $x_n(t_0) = y_n$ existing on $[t_0, t_0 + a]$ such that $\lim_{n \rightarrow \infty} x_n(t) = x(t)$

whenever $\lim_{n \rightarrow \infty} y_n = x_0$.

Then the sequence $\{x_n(t)\}$ converges uniformly to a solution $x(t)$ of (1).

Proof: By the uniform convergence of f_n to f , for $\epsilon = 1$, there exists $N > 0$ such that $\|f_n(t, x) - f(t, x)\| \leq 1$ for all $(t, x) \in [t_0, t_0 + a] \times S[x_0, b]$ and $n \geq N$. Thus $\|f_n(t, x)\| \leq 1 + M$ for all $(t, x) \in [t_0, t_0 + a] \times S[x_0, b]$ and $n \geq N$. For $\epsilon > 0$, let $\delta =$

$\frac{\epsilon}{1+M}$. If $t, s \in [t_0, t_0 + a]$ with $|t - s| < \delta$, then $\|x_n(t) -$

$$x_n(s)\| \leq \left\| \int_s^t \|f_n(w, x_n(w))\| dw \right\| \leq |t - s|(1 + M) < \epsilon. \text{ Therefore the}$$

family $\{x_n(t)\}_{n=N}^{\infty}$ is equicontinuous. Using this fact and the pointwise

convergence of $\{x_n(t)\}$ to $x(t)$, it is easily seen that $x(t)$ is continuous on $[t_0, t_0 + a]$. Thus $x(t)$ is uniformly continuous on $[t_0, t_0 + a]$, and it follows that $x_n(t)$ converges uniformly to $x(t)$ on $[t_0, t_0 + a]$. The standard arguments show that $x(t)$ is a solution of (1). ■

Let us now consider, for each $n \geq 1$, the initial value problem

$$x_n' = f(t, x_n) + \frac{1}{n} r_0, \quad x_n(t_0) = x_0 + \frac{1}{n} r_0 \quad (2)$$

where $r_0 \in K^o$ with $\|r_0\| = 1$.

For convenience we list below some needed hypothesis;

- (H_1) $f(t, x)$ is quasimonotone nondecreasing in x for each $t \in [t_0, t_0 + a]$;
- (H_2) for each $n \geq 1$, a solution of (2) exists on $[t_0, t_0 + a]$;
- (H_3) a solution of (1) exists on $[t_0, t_0 + a]$;
- (H_4) K is a regular cone in E ;
- (H_5) there exists a subsequence $\{x_{n_k}(t)\}$ of $\{x_n(t)\}$ which converges to a function $w(t)$.

The following theorem gives two sets of conditions guaranteeing the existence of the maximal solution of (1) and also a comparison result.

Theorem 3.2: Assume either

- (a) (H_0) , (H_1) , (H_2) , (H_3) , and (H_4) ,
- or (b) (H_0) , (H_1) , (H_2) , and (H_5) .

Then there exists a maximal solution of (1) on $[t_0, t_0 + a]$. Moreover,

if either (a) or (b) holds, and $y \in C[[t_0, t_0 + a], S[x_0, b]]$ and $y'(t) \leq f(t, y(t))$, $y(t_0) \leq x_0$, then $y(t) \leq r(t, t_0, x_0)$ on $[t_0, t_0 + a]$, where $r(t, t_0, x_0)$ is the maximal solution of (1).

Proof: (a) Let $x_n'(t) = f(t, x_n(t)) + \frac{1}{n} r_0$ and $x_{n+1}'(t) = f(t, x_{n+1}(t)) + \frac{1}{n+1} r_0$. Then $x_n'(t) > f(t, x_n(t)) + \frac{1}{n+1} r_0$ and $x_{n+1}'(t) \leq f(t, x_{n+1}(t)) + \frac{1}{n+1} r_0$ with $x_n(t_0) > x_{n+1}(t_0)$. By Theorem 2.1 and the fact that $f(t, x) + \frac{1}{n} r_0$ is quasimonotone nondecreasing in x , $x_n(t) > x_{n+1}(t)$ for all $t \in [t_0, t_0 + a]$.

Suppose that $x(t)$ is a solution of (1) on $[t_0, t_0 + a]$.

Then $x_n'(t) = f(t, x_n(t)) + \frac{1}{n} r_0 > f(t, x_n(t))$ and $x_n(t_0) = x_0 + \frac{1}{n} r_0 > x_0 = x(t_0)$. As before we are lead to $x_n(t) > x(t)$ on $[t_0, t_0 + a]$ for $n \geq 1$. Hence $x_n(t)$ is a decreasing sequence that is bounded below. By the regularity of K , $x_n(t)$ converges. Clearly

$f(t, x_n) + \frac{1}{n} r_0$ converges uniformly to $f(t, x)$ on $[t_0, t_0 + a]$. By

Theorem 3.1, $x_n(t)$ converges uniformly to a solution $r(t)$ of (1).

But $r(t) = \lim_{n \rightarrow \infty} x_n(t) \geq \lim_{n \rightarrow \infty} x(t) = x(t)$ which implies that $r(t, t_0, x_0)$

is the maximal solution of (1) on $[t_0, t_0 + a]$.

(b) The subsequence $\{x_{n_k}(t)\}$ converges to a solution of (1) by Theorem 3.1. That this solution is maximal follows directly as in the proof of part (a).

(c) Consider a solution $x_n(t) = x_n(t, t_0, x_0 + \frac{1}{n} r_0)$ of (2).

Then $x_n'(t) > f(t, x_n(t))$ and $y'(t) \leq f(t, y(t))$ with $x_n(t_0) = x_0 +$

$\frac{1}{n} x_0 > x_0 \geq y(t_0)$. By Theorem 2.1, $y(t) < x_n(t)$. Thus taking the limit either of the sequence used in (a) or the subsequence used in (b), we obtain the result. ■

Note: For convenience we have assumed (H_2) , (H_3) and (H_5) , instead of imposing various kinds of compactness or monotonicity conditions on f which assure these assumptions. For details in this direction see (Refs. 7, 10, 11).

We are now in a position to state our main result of this section concerning the positivity of solutions of (1).

Theorem 3.3: Assume that either (H_0) , (H_1) , (H_2) , (H_3) , and (H_4) , or (H_0) , (H_1) , (H_2) , and (H_5) hold.

(a) If $f(t, x) \geq 0$ for all $(t, x) \in [t_0, t_0 + a] \times S[x_0, b]$, then every solution $x(t)$ of (1) is nondecreasing in t and hence it remains in K for $t \in [t_0, t_0 + a]$ whenever $x_0 \in K$.

(b) If $x(t_0) = x_0 \in K$ and $f(t, 0) = 0$ on $[t_0, t_0 + a]$, then the maximal solution of (1) remains in K .

(c) If solutions of (1) are unique, and $f(t, 0) = 0$ on $[t_0, t_0 + a]$, then $x(t_0) = x_0 \in K$ implies that $x(t, t_0, x_0) \in K$ for all $t \in [t_0, t_0 + a]$.

Proof: (a) Let $t_1 > t_2 > t_0$ and $x(t)$ be a solution of (1).

Then because of (a) we get

$$x(t_1) = x_0 + \int_{t_0}^{t_1} f(s, x(s)) ds = x_0 + \int_{t_0}^{t_2} f(s, x(s)) ds + \int_{t_2}^{t_1} f(s, x(s)) ds \leq x_0 + \int_{t_0}^{t_2} f(s, x(s)) ds = x(t_2),$$

which shows that $x(t)$ is nondecreasing in t . Hence if $x(t_0) \in K$, $\theta \leq x(t_0) \leq x(t)$ for $t \in [t_0, t_0 + a]$.

(b) Let $x_n(t)$ be a solution of (2). Since $f(t, \theta) = \theta$, it follows that $y(t) \equiv \theta$ is a solution of (1) on $[t_0, t_0 + a]$. Hence $x_n'(t) > f(t, x_n(t))$ and $y'(t) = f(t, y(t))$ with $x_n(t_0) = x_0 + \frac{1}{n} r_0 > \theta = y(t_0)$. By Theorem 2.1, $x_n(t) > y(t)$ or $x_n(t) > \theta$ on $[t_0, t_0 + a]$. Thus $r(t, t_0, x_0) \geq \theta$, by Theorem 3.2.

(c) The proof is immediate from (b). ■

The results of Theorem 3.3 extend and generalize some of the corresponding results in finite dimensional situations, see (Ref. 6).

It would appear logical at this point to present a result stating that if the initial value is in the cone, then all solutions remain in the cone. In general, however, this is not the case, as the following trivial example shows.

Example: Let $E = R$, $K = \{x : x \geq 0\}$ and consider $x' = -x^{2/3}$, $x(0) = 0$. Clearly $x \equiv 0$ is the maximal solution. The solution $x(t) = -\left(\frac{t}{3}\right)^3$ is not in the cone for $t > 0$.

4. Quasilinearization.

In this section we will develop the method of quasilinearization for differential equations in abstract cones. First we prove two necessary lemmas.

Lemma 4.1: Let $f \in C[[t_0, t_0 + a] \times D, E]$ and suppose that the Fréchet derivative $f'_x(t, x)$ of f exists for $x \in S[x_0, b]$. If $f(t, x)$ is convex on $S[x_0, b]$ for each $t \in [t_0, t_0 + a]$, then $\eta(t, x, h) \in K$.

Proof: Since the Fréchet derivative of f is assumed to exist, for $\lambda \in (0, 1)$ we have

$$||h|| \eta(t, x, h) = f(t, x + h) - \lambda f(t, x) - (1 - \lambda)f(t, x) - f'_x(t, x)h.$$

The convexity of f assures that $f[t, \lambda x + (1 - \lambda)z] \leq \lambda f(t, x) + (1 - \lambda)f(t, z)$. Setting $z = x + \frac{1}{1 - \lambda}h$ and noting that $x + h = \lambda x + (1 - \lambda)z$, we get $f(t, x + h) \leq \lambda f(t, x) + (1 - \lambda)f(t, z)$. This inequality in turn implies

$$\begin{aligned} ||h|| \eta(t, x, h) &\leq (1 - \lambda)f(t, z) - (1 - \lambda)f(t, x) - f'_x(t, x)h \\ &= (1 - \lambda) \left[f(t, z) - f(t, x) - f'_x(t, x) \frac{h}{1 - \lambda} \right] \\ &= (1 - \lambda) \eta \left(t, x, \frac{h}{1 - \lambda} \right) \left| \frac{h}{1 - \lambda} \right| \\ &= ||h|| \eta \left(t, x, \frac{h}{1 - \lambda} \right). \end{aligned}$$

Thus $\eta(t, x, h) \leq \eta \left(t, x, \frac{h}{1 - \lambda} \right)$, $\lambda \in (0, 1)$. Consequently, setting

$h = \frac{1}{n} y_0$, where $y_0 \in E$ and $\lambda = 1 - \frac{1}{n}$, we obtain $\eta \left(t, x, \frac{1}{n} y_0 \right) \leq$

$\eta(t, x, y_0)$. This relation implies $\theta \leq \eta(t, x, y_0)$, since

$\lim_{n \rightarrow \infty} \eta \left(t, x, \frac{1}{n} y_0 \right) \rightarrow \theta$ and the proof is complete. ■

Lemma 4.2: Let $f \in C[[t_0, t_0 + a] \times D, E]$ and $f(t, x)$ is convex on D for each $t \in [t_0, t_0 + a]$. Suppose that the Fréchet derivative

$f_x(t, x)$ of f exists for $x \in S[x_0, b]$. For each $t \in [t_0, t_0 + a]$, if $f(t, x)$ is quasimonotone nondecreasing in x , then the function

$$G(t, v) \equiv f(t, \alpha(t)) + f_x(t, \alpha(t))(v - \alpha(t))$$

is quasimonotone nondecreasing in v for each $t \in [t_0, t_0 + a]$, where $\alpha \in C[[t_0, t_0 + a], S[x_0, b]]$.

Proof: Let $y, z \in D$ be such that $y - z + \alpha(t) \in D$. Suppose that $y \leq z$, $c \in K^*$, and $cy = cz$. Then $y - z + \alpha(t) \leq \alpha(t)$ and $c[y - z + \alpha(t)] = c[\alpha(t)]$. The quasimonotonicity of f implies

$$cf(t, y - z + \alpha(t)) \leq cf(t, \alpha(t)). \quad (3)$$

From the definition of the Fréchet derivative, it follows that

$$\begin{aligned} cf(t, y - z + \alpha(t)) - cf(t, \alpha(t)) &= c[f_x(t, \alpha(t))(y - z)] \\ &+ ||y - z|| \eta(t, x, y - z). \end{aligned}$$

Since by Lemma 4.1, $\eta(t, x, y - z) \in K$, the relation (3) yields

$$c[f_x(t, \alpha(t))(y - z)] \leq 0$$

or
$$c[f_x(t, \alpha(t))y] \leq c[f_x(t, \alpha(t))z].$$

This last inequality implies that $G(t, v)$ is quasimonotone nondecreasing in v for each $t \in [t_0, t_0 + a]$. ■

We now state the main result of this section concerning the method of quasilinearization.

Theorem 4.1: Assume that (H_0) , (H_1) , and (H_4) hold. Suppose further

that

(H₆) $f_x(t, x)$ exists and is continuous for $(t, x) \in [t_0, t_0 + a] \times S[x_0, b]$ and $\|f_x(t, x)\| \leq L$ for $(t, x) \in [t_0, t_0 + a] \times S[x_0, b]$, where a is chosen such that $a(M + Lb) \leq b$; and

(H₇) for each $t \in [t_0, t_0 + a]$, $f(t, x)$ is convex on $S[x_0, b]$. Then there exists a sequence of functions $\{z_n(t)\}$ on $[t_0, t_0 + a]$ such that

(a) for each $n \geq 1$ and $t \in [t_0, t_0 + a]$, $z_n(t)$ is the solution of the linear differential equation

$$z' = f(t, z_{n-1}) + f_x(t, z_{n-1})(z - z_{n-1}), \quad z(t_0) = x_0,$$

and $z_n(t) \leq x(t)$, where $x(t)$ is a solution of (1);

(b) $\lim_{n \rightarrow \infty} z_n(t) = x(t)$ uniformly on $[t_0, t_0 + a]$.

Proof: By (H₆) it is easy to conclude that $f(t, x)$ satisfies a Lipschitz condition in x for $(t, x) \in [t_0, t_0 + a] \times S[x_0, b]$, and consequently there exists a unique solution $x(t)$ for (1) on $[t_0, t_0 + a]$. Furthermore, for each $\alpha \in C[[t_0, t_0 + a], S[x_0, b]]$, the solution $y_\alpha(t)$ of $y' = f(t, \alpha(t)) + f_x(t, \alpha(t))(y - \alpha(t))$, $y(t_0) = x_0$, exists on $[t_0, t_0 + a]$ and $y_\alpha(t) \in S[x_0, b]$, for $t \in [t_0, t_0 + a]$.

We now define the sequence $\{z_n(t)\}$ as follows. Choose $z_0(t) = x_0$ and let $z_1(t)$ be the solution of

$$z' = f(t, x_0) + f_x(t, x_0)(z - x_0), \quad z(t_0) = x_0,$$

on $[t_0, t_0 + a]$ such that $z_1(t) \in S[x_0, b]$. This way we can define successively $\{z_n(t)\}$ as the solutions of

$$z' = f(t, z_{n-1}(t)) + f_x(t, z_{n-1}(t))(z - z_{n-1}(t)), \quad z(t_0) = x_0. \quad (4)$$

Next we shall show that $\{z_n(t)\}$ is a monotone nondecreasing sequence and that $z_n(t) \leq x(t)$, on $[t_0, t_0 + \alpha]$, for each $n \geq 1$. For $t \in [t_0, t_0 + \alpha]$ and for $n \geq 1$, we have

$$\begin{aligned} z_n'(t) &= f(t, z_{n-1}(t)) + f_x(t, z_{n-1}(t))(z_n(t) - z_{n-1}(t)) \\ &\leq f(t, z_n(t)), \end{aligned}$$

which yields by Theorem 3.2 that

$$z_n(t) \leq x(t), \quad t \in [t_0, t_0 + \alpha].$$

To show that $z_{n-1}(t) \leq z_n(t)$ for each $n \geq 1$, consider

$$\begin{aligned} z_{n-1}'(t) &= f(t, z_{n-2}(t)) + f_x(t, z_{n-2}(t))[z_{n-1}(t) - z_{n-2}(t)] \\ &\leq f(t, z_{n-1}(t)) \\ &= f(t, z_{n-1}(t)) + f_x(t, z_{n-1}(t))[z_{n-1}(t) - z_{n-1}(t)]. \end{aligned}$$

This implies by Theorem 3.2 that $z_{n-1}(t) \leq z_n(t)$ on $[t_0, t_0 + \alpha]$, proving that $\{z_n(t)\}$ is nondecreasing. Thus by (H_4) , it follows that

$$\lim_{n \rightarrow \infty} z_n(t) = z(t) \quad \text{uniformly on } [t_0, t_0 + \alpha].$$

Using the fact that $z_n(t)$ is a solution of (4), we obtain

$$\begin{aligned} &||z_n(t) - z_n(t_0) - \int_{t_0}^t f(s, z_{n-1}(s))ds|| \\ &= ||\int_{t_0}^t f_x(s, z_{n-1}(s))(z_n(s) - z_{n-1}(s))ds|| \\ &\leq \left| \int_{t_0}^t ||f_x(s, z_{n-1}(s))(z_n(s) - z_{n-1}(s))|| ds \right| \end{aligned}$$

$$\leq \left| \int_{t_0}^t L \|z_n(s) - z_{n-1}(s)\| ds \right|.$$

Since L is independent of n , taking limits as $n \rightarrow \infty$ we obtain

$$\|z(t) - x_0 - \int_{t_0}^t f(s, z(s)) ds\| \leq 0,$$

so that $z(t) = x(t)$, the unique solution of (1). ■

It is easily seen that if K is a regular cone, then $\tilde{K} = \{x : -x \in K\}$ is a regular cone; and if f is quasimonotone nondecreasing with respect to K , it is also quasimonotone nondecreasing with respect to $\tilde{K}$. Finally, observe that if $x < y$ with respect to K , then $x > y$ with respect to $\tilde{K}$. Thus if f is concave with respect to K , it is convex with respect to $\tilde{K}$; and a lower bound for a function with respect to $\tilde{K}$ is an upper bound for that function with respect to K . These observations lead to the following result.

Theorem 4.2: Assume that (H_0) , (H_1) , (H_4) , and (H_6) hold. In addition assume that

(H_7') for each $t \in [t_0, t_0 + a]$, $f(t, x)$ is concave on $S[x_0, b]$.

Then there exists a sequence of functions $\{\tilde{z}_n(t)\}$ on $[t_0, t_0 + a]$ such that

(a) for each $n \geq 1$ and $t \in [t_0, t_0 + a]$, $\tilde{z}_n(t)$ is the solution of the linear differential equation

$$\tilde{z}' = f(t, \tilde{z}_{n-1}) + f_x(t, \tilde{z}_{n-1})(\tilde{z} - \tilde{z}_{n-1}), \quad \tilde{z}(t_0) = x_0,$$

and $\tilde{z}_n(t) \geq x(t)$, where $x(t)$ is a solution of (1);

$$(b) \lim_{n \rightarrow \infty} \tilde{x}_n(t) \rightarrow x(t) \text{ uniformly on } [t_0, t_0 + a].$$

Proof indication: The proof is accomplished by reinterpreting all cone inequalities in $\tilde{K}$ and applying Theorem 4.1. ■

5. Applications.

In this section we will prove some results showing applications of the method of quasilinearization as well as the flexibility of dealing with this method through the use of cones in Banach spaces. As we indicated in the introduction it is frequently more useful to apply the method of quasilinearization to the comparison equation rather than the original equation. The seemingly strong restriction of dealing with a regular cone that has nonvoid interior becomes much more natural when we realize the flexibility of choosing the space in which the comparison equation assumes values.

To this end we consider a real Banach space E , a regular cone $K \subset E$, and a real linear space X . If there exists $||\cdot||_K : X \rightarrow K$ satisfying

- i) $||x||_K = \theta$ if and only if $x = \theta$,
- ii) $||\alpha x||_K = |\alpha| ||x||_K$, and
- iii) $||x + y||_K \leq ||x||_K + ||y||_K$,

then X is said to be a K -normed linear space. If X is complete with respect to $||\cdot||_K$, then X is said to be a K -Banach space. For results dealing with existence of solutions in K -Banach spaces, see (Ref. 12). Since Theorem 2.1 can be proved with only minor modifications for D_+ instead of total derivatives, the results of the comparison Theorem 3.2 also hold for D_+ .

Theorem 5.1: Let $X, ||\cdot||_K$ be a K -Banach space where K is a regular cone in a real Banach space E . Assume that

(1) $f \in C[[t_0, t_0 + \alpha] \times E, E]$ and f satisfies (H_0) , (H_1) , (H_2) , (H_3) , and (H_4) , or (H_0) , (H_1) , (H_2) , and (H_5) ,

(2) $F \in C[[t_0, t_0 + \alpha] \times X, X]$ and for each $x_0 \in X$ there exists a solution $x(t, t_0, x_0)$ of $x' = F(t, x)$, $x(t_0) = x_0$, on $[t_0, t_0 + \alpha]$,

(3) $\lim_{h \rightarrow 0^+} \frac{1}{h} \left(||x + hF(t, x)||_K - ||x||_K \right) \leq f(t, ||x||_K)$

for each $(t, x) \in [t_0, t_0 + \alpha] \times E$.

Then for any solution $x(t, t_0, x_0)$ of $x' = F(t, x)$, $x(t_0) = x_0$, we have $||x(t, t_0, x_0)||_K \leq r(t, t_0, ||x_0||_K)$ on $[t_0, t_0 + \alpha]$ where $r(t, t_0, ||x_0||_K)$ is the maximal solution of $u' = f(t, u)$, $u(t_0) = x_0$.

Proof: Let $m(t) = ||x(t, t_0, x_0)||_K$ and by hypothesis (3) we see that $D_+ m(t) \leq g(t, m(t))$ and $m(t_0) = ||x_0||_K$. Thus from Theorem 3.2 we conclude $||x(t, t_0, x_0)||_K = m(t) \leq r(t, t_0, ||x_0||_K)$ on $[t_0, t_0 + \alpha]$. ■

We can now combine the results of Theorem 5.1 with the method of quasilinearization in Theorem 4.2 and obtain the following theorem.

Theorem 5.2: Assume the hypothesis of Theorem 5.1 and in addition assume that f satisfies the hypothesis of Theorem 4.2. If

$x(t, t_0, x_0)$ is a solution of $x' = F(t, x)$, $x(t_0) = x_0$, then the sequence of functions $\{\tilde{a}_n(t)\}$ assured by Theorem 4.2 forms of monotonic nonincreasing sequence of upper bounds for $||x(t, t_0, x_0)||_K$.

Proof: By Theorem 5.1 $r(t, t_0, ||x_0||_K) \geq ||x(t, t_0, x_0)||_K$ and by Theorem 4.2, $\{\tilde{a}_n(t)\}$ is a monotonic nonincreasing sequence which converges to $r(t, t_0, x_0)$. ■

Example 5.1: If X is itself a Banach space, $E = R$, $K = R_+$, and $\|x\|_K = \|x\|$, then the method of quasilinearization applied to the comparison equation is precisely the usual method of quasilinearization but yields bounds for the norm of the solution of the differential equation.

Example 5.2: If X is a linear space, $E = R^n$, $K = R_+^n$, and $\|x\|_K = \|x\|_G$ (see (Refs. 13, 14) for information concerning generalized norms) then the method of quasilinearization applied to the comparison equation yields bounds for the generalized norm of the solution of the differential equation.

Examples 5.1 and 5.2 indicate the flexibility of choosing the comparison equations in "nice" spaces where known cones are positive, closed, regular, and have non-void interior. The next theorem indicates another application of the method of quasilinearization applied to a comparison equation in an arbitrary Banach space in which we deal with a regular cone and actually compare within the same space to obtain an upper bound for a delay differential equation in a Banach space. We use the following notation, see (Ref. 4).

Let $T > 0$ and define $E_0 = C[[-T, 0], E]$. The norm of $\phi \in E_0$ is the sup norm. For $y \in C[[-T, \infty), E]$ and $t \geq 0$, the notation y_t means $y_t \in E_0$ defined by $y_t(s) = y(t + s)$ for all $s \in [-T, 0]$. Let $\Lambda = \{\phi \in E_0 \mid \phi(s) \leq \phi(0) \text{ for all } s \in [-T, 0]\}$.

Lemma 5.1: Let $f \in C[[t_0, t_0 + \alpha] \times E, K]$ and suppose further that (H_1) , (H_2) , (H_3) with the added hypothesis of uniqueness, and (H_4) hold. In addition, assume that

(1) $F \in C[[t_0 - T, t_0 + \alpha] \times E_0, E]$;

(2) if $\phi \in \Lambda$ and $c \in K^*$ such that $c\phi(s) \leq c\phi(0)$ on $[-T, 0]$, then $cF(t, \phi) \leq cf(t, \phi(0))$.

If $\phi_0 \in E_0$, $y(t, \phi_0)$ is a solution of $y' = F(t, y_t)$, $y_{t_0} = \phi_0$ and $x(t, t_0, x_0)$ is a solution of $x' = f(t, x)$, $x(t_0) = x_0$, then $y(t) \leq x(t, t_0, x_0)$ on $[t_0, t_0 + \alpha]$.

Proof: Consider $x' = f(t, x) + \frac{1}{n} r_0$ with $x(t_0) = x_0 + \frac{1}{n} r_0$. Assume that there exists $t_1 > t_0$ such that $y(t) < x_n(t)$ on $[t_0, t_1]$, $y(t_1) \leq x_n(t_1)$, but $y(t_1) \neq x_n(t_1)$. That is, $x_n(t_1) - y(t_1)$ is on the boundary of K . By the Ascoli-Mazur Theorem there exists $c \in K^*$ such that $cy(t_1) = cx_n(t_1)$. Thus we have $y(t) \leq x_n(t)$ on $[t_0, t_1]$. Since $f(t, x_n) \in K$, by Theorem 3.3(a), $x_n(t)$ is nondecreasing and thus $y(t) \leq x_n(t) \leq x_n(t_1)$ on $[t_0, t_1]$.

Define $\phi = y_{t_1}$. Thus $\phi(s) = y_{t_1}(s) = y(s + t_1)$; but $s + t_1 < t_1$, so $\phi(s) \leq x_n(t_1)$. Hence $c\phi(s) \leq cx_n(t_1) = cy(t_1) = cy_{t_1}(0) = c\phi(0)$. By hypothesis (2) $F(t, \phi) \leq f(t, \phi(0))$, so that

$$cF(t, \phi) \leq cf(t, \phi(0)). \quad (5)$$

Direct computation shows that $cF(t_1, y_{t_1}) \geq c[f(t_1, x_n(t_1)) + \frac{1}{n} r_0]$. By the quasimonotonicity of f we obtain $cf(t_1, y(t_1)) \leq cf(t_1, x_n(t_1))$ so that $cF(t_1, y_{t_1}) \geq c[f(t_1, y(t_1)) + \frac{1}{n} r_0]$. But $\phi = y_{t_1}$ and so $y(t_1) = \phi(0)$, thus

$$cF(t_1, \phi) \geq c[f(t_1, \phi(0)) + \frac{1}{n} r_0]$$

which contradicts (5).

Thus $y(t) < x_n(t)$ for $t > t_0$. But $\lim_{n \rightarrow \infty} x_n(t) = x(t, t_0, y_0)$ by Theorem 3.2 so that $y(t) \leq x(t)$ for $t > t_0$. ■

Now combining Lemma 5.1 and Theorem 4.2 we obtain an upper bound for the solution of a delay equation.

Theorem 5.3: Assume the hypothesis of Lemma 5.1 and in addition assume that f satisfies the conditions of Theorem 4.2. If $y' = F(t, y_t)$, $y_{t_0} = \phi_0$, and $x' = f(t, x)$, $x(t_0) = x_0$, then the sequence of functions $\{\tilde{z}_n(t)\}$ assured by Theorem 4.2 forms a monotonic non-increasing sequence of upper bounds for $y(t, \phi_0)$ a solution of $y' = F(t, y_t)$, $y_{t_0} = \phi_0$.

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