ON THE CONSTRUCTION OF A NORM
ASSOCIATED WITH THE MEASURE OF NONCOMPACTNESS

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ABSTRACT

It is shown that the bounded, nonempty subsets of a reflexive Banach space E can be imbedded in another Banach space $B(E)$ in such a manner so that the measure of noncompactness corresponds to the norm in $B(E)$. The results are applied to ordinary differential equations theory.

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1. Introduction.

Existence theorems for ordinary differential equations in an infinite dimensional Banach space E has undergone an extensive study in recent years. One approach, which is used to prove the existence of a solution to the Cauchy problem.

$$(E) \quad x'(t) = f(t, x), \quad x(t_0) = x_0,$$

requires a condition which is not needed in finite dimensional spaces.

Let α be a measure of noncompactness, then one imposes a condition like

$$(C) \quad \alpha(f(t, U)) \leq g(t, \alpha(U))$$

where U is any bounded subset of E and $g: R^+ \times R^+ \rightarrow R^+$ is continuous and satisfies the condition that $g(t, 0) = 0$ and the only solution of

$$(S) \quad u' = g(t, u), \quad u(0) = 0$$

is $u(t) \equiv 0$. Several references are given in [3, p. 467]. This can be considered as an example where α is treated as if it were a norm on the bounded subsets of E .

In this paper we show that, in fact, α is a norm in the sense that there is semi-linear, continuous map q of the bounded subsets of E into a Banach space $Q(E)$ with norm $||\cdot||$ such that $||q(A)|| = \alpha(A)$.

In this manner we are able to construct an α topology on the bounded subsets of E which allows us to obtain existence, uniqueness and comparison results of differential equations in the α topology. In this setting $f(t, \cdot)$ maps bounded subsets of E into bounded subsets of E .

2. Preliminaries.

Let E be a real Banach space. Denote by $B(E)$ (resp. $C(E)$, $C_0(E)$, $K(E)$, $K_0(E)$) the family of all non-void bounded (resp. bounded closed, bounded closed convex, pre-compact, compact convex) subsets of E , $\bar{A}$ the closure, $\overline{co} A$ the closed convex hull of $A \subset E$. Let $A, B \in B(E)$. Define the Hausdorff distance:

$$d[A, B] = \inf \{t > 0 \mid A \subset B + tS, B \subset A + tS\}$$

where S is the unit sphere in E . It is known [5] that the family $C(E)$ and, therefore, its subfamilies $C_0(E)$, $K_0(E)$ are metrized by d and that d is a semi-metric over $B(E)$.

Addition of sets is defined by

$$A + B = \{a + b \mid a \in A, b \in B\},$$

and scalar multiplication by

$$\lambda A = \{\lambda a \mid a \in A\}.$$

The additive unit ϕ is then the singleton set containing the zero element of E and $-A = -1A$. Note $A - A \supset \phi$ but the reverse inclusion is not true in general. Thus $B(E)$ is not a linear space. Radstrom [5] has shown that because the law of cancellation holds for convex sets, appropriate

subclasses may be embedded in normed linear spaces. The following lemma is an immediate corollary of Radstrom's result.

LEMMA 2.1. Let E be a real reflexive Banach space. Then $C_0(E)$ (resp. $K_0(E)$) can be embedded as a closed convex cone in a real Banach space $BC_0(E)$ (resp. $BK_0(E)$) in such a way that

- a) the embedding is isometric,
- b) addition in $BC_0(E)$ induces addition in $C_0(E)$,
- c) multiplication by non-negative scalars in $BC_0(E)$ induces the corresponding operation in $C_0(E)$,
- d) $BK_0(E)$ is a subspace of $BC_0(E)$,
- e) Let $i: C_0(E) \rightarrow BC_0(E)$ denote the embedding map, then the linear manifold $\{i(A) - i(B) \mid A, B \in C_0(E)\}$ (resp. $\{i(A) - i(B) \mid A, B \in K_0(E)\}$) is dense in $BC_0(E)$ (resp. $BK_0(E)$).

The measure of noncompactness, $\alpha(A)$, of a bounded subset A is defined by

$$\alpha(A) = \inf\{t > 0 \mid \text{there exists } C \in K(E) \text{ such that}$$

$A \subset C + tS\}$. We will show (see Lemma 2.3) that α is equivalent to Kuratowski's measure of noncompactness [2].

LEMMA 2.2. (De Blasi [1]). The functional α has the properties:

- a) $A \subset B$ implies $\alpha(A) \leq \alpha(B)$
- b) $\alpha(A) = \alpha(\overline{A})$.

- c) $\alpha(A) = 0$ if and only if $\bar{A}$ is compact
- d) $\alpha(A + B) \leq \alpha(A) + \alpha(B)$
- e) $\alpha(tA) = t\alpha(A), \quad t \geq 0$
- f) $\alpha(A) = \alpha(\overline{\text{co}} A)$
- g) $\alpha\left(\bigcup_{t \in [0, s]} tA\right) = s\alpha(A)$
- h) $\alpha(S) \leq 1, \quad \alpha(S) = 1$ if $\dim(E) = \infty$

LEMMA 2.3. Let E be a real Banach space. For $A \in B(E)$ define:

- 1. (Kuratowski) $\gamma(A) = \inf\{t > 0 \mid A \text{ can be covered by a finite number of sets of diameter } \leq t\}$.
- 2. $v(A) = \inf\{t > 0 \mid A \text{ can be covered by a finite number of spheres of radius } \leq t\}$.
- 3. $\mu(A) = \inf\{d[A, C] \mid C \in K(E)\}$.

Then

- a) $\alpha(A) = v(A) = \mu(A)$
- b) $\alpha(A) \leq \gamma(A) \leq 2\alpha(A)$

PROOF. We require several inequalities.

(i) $\alpha(A) \leq \mu(A)$. Let $t > \mu(A)$. Then there exists $C \in K(E)$ such that $A \subset C + tS$. Thus, by the properties of α , $\alpha(A) \leq \alpha(C + tS) \leq t$. Since $t > \mu(A)$ is arbitrary, $\alpha(A) \leq \mu(A)$.

(ii) $\mu(A) \leq \alpha(A)$. Let $t > \alpha(A)$. There exists $C \in K(E)$ such that $A \subset C + tS$. Let $D = \{x \mid x \in C, \inf\{\|x - a\| \mid a \in A\} \leq t\}$ where $\|\cdot\|$ denotes the norm in E . Then $D \in K(E)$, $A \subset D + tS$, and

$D \subset A + tS$. Thus $d[A, D] \leq t$. We conclude that $\mu(A) \leq t$. Since $t > \alpha(A)$ is arbitrary $\mu(A) \leq \alpha(A)$.

(iii) $\alpha(A) \leq v(A)$. Let $t > v(A)$. Then there exists a set C consisting of a finite number of points such that $A \subset C + tS$. Since $C \in K(E)$, $\alpha(A) \leq t$. Since $t > v(A)$ is arbitrary, $\alpha(A) \leq v(A)$.

(iv) $v(A) \leq \alpha(A)$. Suppose $t > \alpha(A)$. Then there exists $C \in K(E)$ such that $A \subset C + tS$. Let $r > 0$. Since C is pre-compact there is a finite set D such that $C \subset D + rS$. Thus $A \subset D + rS + tS = D + (r + t)S$. Since $r > 0$ and $t > \alpha(A)$ are arbitrary, $v(A) \leq \alpha(A)$.

(v) $\alpha(A) \leq \gamma(A)$. Let $t > \gamma(A)$. Then there exists a set C consisting of a finite number of points such that $A \subset C + tS$. It then follows (as in (iii)) that $\alpha(A) \leq \gamma(A)$.

(vi) $\gamma(A) \leq 2v(A)$. This is immediate from the definitions since in one case the infimum is taken over coverings which are limited to spheres. The factor 2 occurs because in one case we use the diameter while in the other case we use the radius.

From (i) and (ii), we have $\alpha(A) = \mu(A)$. From (iii) and (iv) we have $\alpha(A) = v(A)$. Finally, from (v) and (vi) we have $\alpha(A) \leq \gamma(A) \leq 2v(A) = 2\alpha(A)$.

3. The α -topology.

We assume throughout that E is a real reflexive Banach space.

LEMMA 3.1. Let $A, B \in B(E)$, $a, b \geq 0$, then $\overline{co}(aA + bB) = a \overline{co}(A) + b \overline{co}(B)$.

PROOF. Since $\overline{co}(aA) = a \overline{co}(A)$ and $\overline{co}(bB) = b \overline{co}(B)$, it suffices to show that $\overline{co}(A + B) = \overline{co}(A) + \overline{co}(B)$. Since $\overline{co}(A) + \overline{co}(B)$ is closed and convex (see [5]) $\overline{co}(A + B) \subset \overline{co}(A) + \overline{co}(B)$. We now prove the reverse inclusion. Clearly $\overline{co} A + \overline{co} B \subset \overline{co} A + \overline{co} B$. Thus it remains to show that $\overline{co}(A) + \overline{co}(B) \subset \overline{co}(A + B)$. But this is an immediate consequence of the definition of convex hull.

Let $A, B \in B(E)$. We define

$$(3.1) \quad \overline{\alpha}[A, B] = \inf\{d[\overline{co} A + K_1, \overline{co} B + K_2], K_1, K_2 \in K_0(E)\}$$

$$\text{LEMMA 3.2.} \quad \overline{\alpha}[A, B] = \inf\{d[\overline{co}(A) + C_1, \overline{co}(B) + C_2] \mid C_1, C_2 \in K(E)\}.$$

PROOF. Let $\overline{\beta}[A, B]$ denote the infimum over $K(E)$. Since $K_0(E) \subset K(E)$, $\overline{\beta}[A, B] \leq \overline{\alpha}[A, B]$. To obtain the reverse inequality, suppose $\overline{\beta}[A, B] < t$. Then there exist $C_1, C_2 \in K(E)$ such that $\overline{co}(A) + C_1 \subset \overline{co}(B) + C_2 + tS$, $\overline{co}(B) + C_2 \subset \overline{co}(A) + C_1 + tS$. Applying Lemma 3.1, $\overline{co}(A) + \overline{co}(C_1) \subset \overline{co}(B) + \overline{co}(C_2) + tS$ and $\overline{co}(B) + \overline{co}(C_2) \subset \overline{co}(A) + \overline{co}(C_1) + tS$. Thus obtain $\overline{\alpha}[A, B] \leq \overline{\beta}[A, B]$ to conclude the argument.

THEOREM 3.1. There exists a continuous map q from the space $(B(E), d[\cdot, \cdot])$ onto a closed convex cone $\kappa(E)$ of a Banach space $(Q(E), \|\cdot\|)$ satisfying:

- a) $q(aA + bB) = aq(A) + bq(B)$, $a \geq 0$, $b \geq 0$
- b) $\|q(A) - q(B)\| = \overline{\alpha}[A, B]$
- c) $\|q(A)\| = \alpha(A)$
- d) $q(\overline{co}(A)) = q(A)$.

PROOF. Let $B C_0(E)$ and $B K_0(E)$ be as in Lemma 2.1. Define $Q(E)$ as the quotient space $B C_0(E)/B K_0(E)$. Let i be the embedding map $C_0(E) \rightarrow B C_0(E)$, p the projection map $B C_0(E) \rightarrow Q(E)$, and let $m: B(E) \rightarrow C_0(E)$ be defined by $m(U) = \overline{co}(U)$. Define $q = pim$. Since all the three maps p , i , and m have the semi-linearity property a), so does q .

Property d) is obvious from the definition.

For property b), let $\|\cdot\|_0$ denote the norm in $B C_0(E)$.

Then $\|q(A) - q(B)\| = \inf\{\|q(A) + q(K_1) - q(B) - q(K_2)\|_0, K_1, K_2 \in K_0(E)\} = \inf\{\|q(A + K_1) - q(B + K_2)\|_0 | K_1, K_2 \in K_0(E)\} = \inf\{d[\overline{co}(A) + K_1, \overline{co}(B) + K_2] | K_1, K_2 \in K_0(E)\} = \overline{\alpha}[A, B]$.

To obtain c) suppose $\|q(A)\| < t$. Then there exists $K_1, K_2 \in K_0(E)$ such that $\overline{co}(A) + K_1 \subset K_2 + tS$. Thus by the properties of α (see Lemma 2.2) $\alpha(A) = \alpha(\overline{co}(A)) \leq \alpha(K_2 + tS) \leq t$. Since $t > \|q(A)\|$ is arbitrary $\alpha(A) \leq \|q(A)\|$. For the reverse inequality, note that $\|q(A)\| = \overline{\alpha}[A, \Phi] = \inf\{d[\overline{co}(A) + C_1, C_2] | C_1, C_2 \in K(E)\} \leq \inf\{d[\overline{co}(A), C] | C \in K(E)\} = \mu(\overline{co}(A)) = \alpha(\overline{co}(A)) = \alpha(A)$ where we have used Lemmas 2.3 and 3.2. This completes the argument.

LEMMA 3.3. $\overline{\alpha}$ has the following properties:

- a) $\overline{\alpha}[A, B] = 0$ when $A = B$,
- b) $0 \leq \overline{\alpha}[A, B] = \overline{\alpha}[B, A] \leq d[A, B]$,
- c) $\overline{\alpha}[tA, tB] = t\overline{\alpha}[A, B]$, $t \geq 0$,
- d) $\overline{\alpha}[A + B, C + D] \leq \overline{\alpha}[A, C] + \overline{\alpha}[B, D]$,

- e) $|\bar{\alpha}[A,B] - \bar{\alpha}[C,D]| \leq \bar{\alpha}[A,C] + \bar{\alpha}[B,D],$
- f) $|\alpha(A) - \alpha(B)| \leq \bar{\alpha}[A,B],$
- g) $\bar{\alpha}[A,K] = \alpha(A)$ when $K \in K(E),$
- h) $\bar{\alpha}[A,B] = \alpha(A - B)$ when $\bar{\alpha}[A,B + (A - B)] = 0$

PROOF. These properties follow from Theorem 3.1 which establishes the relationship between α and $\bar{\alpha}$ to a norm. We verify d), g), and h) to illustrate the method of proof.

$$\begin{aligned} \text{d) } \bar{\alpha}[A + B, C + D] &= ||q(A + B) - q(C + D)|| \\ &= ||(q(A) - q(C) + q(B) + q(D))|| \leq ||q(A) - q(C)|| \\ &\quad + ||q(B) - q(D)|| = \bar{\alpha}[A,C] + \bar{\alpha}[B,D] \end{aligned}$$

g) Let $K \in K(E)$. Then $||q(K)|| = \alpha(K) = 0$. Thus $q(K) = 0$. So $\bar{\alpha}[A,K] = ||q(A) - q(K)|| = ||q(A)|| = \alpha(A)$.

h) Suppose $\bar{\alpha}[A,B + (A - B)] = 0$. Then $q(A) = q(B + (A - B)) = q(B) + q(A - B)$. Thus $\bar{\alpha}[A,B] = ||q(A) - q(B)|| = ||q(A - B)|| = \alpha(A - B)$.

Thus completes the proof.

REMARK. If $B(E)$ were a linear space then we would always have $A = B + (A - B)$, in which case $\bar{\alpha}[A,B]$ and $\alpha(A - B)$ would be identical measures. There are, however, cases in which $A \neq B + (A - B)$. In fact this occurs whenever $A = B + W$ for some $W \in B(E)$.

4. Applications to differential equations.

Let E be a real reflexive Banach space. We let $S(X_0, b) = \{X \in B(E) : \bar{\alpha}[X, X_0] \leq b\}$. We denote by J either a bounded interval $[t_0, t_0 + a]$ or an unbounded interval $[t_0, \infty)$. Let $R_0 = J \times S(X_0, b)$ and let R^+ denote the nonnegative real line. Let $g \in C[J \times R^+, R]$. We say that a map $f: J \times B(E) \rightarrow B(E)$ is g -comparable with respect to α if

$$(C_\alpha) \quad \limsup_{h \rightarrow 0} \frac{1}{h} \bar{\alpha}[X + h f(t, X), Y + h f(t, Y)] - \bar{\alpha}[X, Y] \leq g(t, \bar{\alpha}[X, Y])$$

Note that this condition is satisfied when one assumes Perron's type uniqueness condition, namely,

$$(P_\alpha) \quad \bar{\alpha}[f(t, X), f(t, Y)] \leq g(t, \bar{\alpha}[X, Y]).$$

One can also work with a condition more general than (C_α) (see [3], p. 461).

Let $X, Y \in C[J, B(E)]$. We say that Y is an α derivative of X and write $Y = X'$ if

$$\lim_{h \rightarrow 0} \frac{1}{h} \bar{\alpha}[X(t+h), X(t) + h Y(t)] = 0, \quad t \in J$$

The following result deals with existence of solutions to the abstract Cauchy problem

$$(E_\alpha) \quad X' = f(t, X), \quad X(t_0) = X_0 \in B(E),$$

THEOREM 4.1. Assume that

i) $f \in C[R_0, B(E)]$ (where continuity in $B(E)$ is with respect to α) and $a > 0$, $b > 0$ and $M \geq 1$ are chosen so that $\alpha(f(t, X)) \leq M$ on R_0 ,

ii) $g \in C[J \times R^+, R]$, $g(t, 0) \equiv 0$ and $u \equiv 0$ is the unique solution of $u' = g(t, u)$, $u(t_0) = 0$,

iii) f is g -comparable with respect to $\bar{\alpha}$ for $t \in J$, $X, Y \in B(X_0, b)$. Then there exists an α -solution $X(t, t_0, X_0)$ to (E_α) which is unique in the sense that if $Y(t) = y(t, t_0, X_0)$ is also an α -solution then $\bar{\alpha}[Y(t), X(t, t_0, X_0)] \equiv 0$.

PROOF. We apply Theorem 3.1. Define $\hat{f}: J \times q(S(X_0, b)) \rightarrow q(S(X_0, b))$ by $\hat{f}(t, x) = q(\overline{co}) f(t, q^{-1}(x))$. Note that $\hat{f}$ is well defined, for if $q(A) = q(B)$, then $\bar{\alpha}[A, B] = 0$ and hence $\bar{\alpha}[f(t, A), f(t, B)] \leq g(t, 0) = 0$. Thus $q(f(t, A)) = q(f(t, B))$.

Define $Q_0 = J \times q(S(X_0, b))$. Then $\hat{f} \in C[Q_0, Q]$, $\|f(t, x)\| \leq M$ on Q_0 and $\limsup_{h \rightarrow 0} \frac{1}{h} [\|x - y + h[\hat{f}(t, x) - \hat{f}(t, y)]\| - \|x - y\|] \leq g(t, \|x - y\|)$, for $t \in J$, $x, y \in q(S(X_0, b)) = \{x \in \kappa(E) : \|x - q(X_0)\| \leq b\}$.

It follows from Theorem 1 of [3] that there exists a unique solution $z(t) \equiv x(t, t_0, q(X_0))$ to the Cauchy problem

$$(qE_\alpha) \quad x' = \hat{f}(t, x), \quad x(t_0) = q(X_0)$$

Since $z(t) \in \kappa(E) = q(B(E))$ there exists $A(t) \in B(E)$ such that $q(A(t)) = z(t)$. By Theorem 3.1, $\bar{\alpha}[A(t+h), A(t) + h\hat{f}(t, A(t))] = \|[z(t+h) - z(t) - h\hat{f}(t, z(t))]\| = o(h)$ and $\bar{\alpha}[A(t), X_0] = \|z(t) - q(X_0)\| \rightarrow 0$ as $t \rightarrow 0$. Thus $X(t, t_0, X_0) = A(t)$ is an α -solution.

If $Y(t)$ is also an α -solution then $q(Y(t))$ is a solution to (qE_α) . Since the solution of (qE_α) is unique, $q(Y(t)) = q(X(t, t_0, X_0))$ which means that $\bar{\alpha}[Y(t), X(t, t_0, X_0)] \equiv 0$.

This completes the proof.

REMARK. Using the theory of differential inequalities [4] one can obtain comparison estimates such as $\bar{\alpha}[X(t, t_0, X_0), X(t, t_0, Y_0)] \leq \Gamma(t, t_0, [X_0, Y_0])$ where $\Gamma(t, t_0, u_0)$ is the maximal solution of $u' = g(t, u)$, $u(t_0) = u_0$. In later work, we plan to use this type of estimate to obtain stability results.

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