STRUCTURAL INFERENCE ON RELIABILITY
IN A LOGNORMAL MODEL

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ABSTRACT.

The theory of structural inference, as developed by Fraser (1968), is based on a group-theoretic approach using invariant Haar measures to Fisher's fiducial theory. Structural inference theory constructs a unique distribution, conditional on the given sample information only, for the parameters of a measurement model. Based on the structural density for the two-parameter lognormal distribution, the structural density and distribution function for the reliability function are derived. Consequently, expressions for structural point and interval estimates of the reliability function are developed. Approximations for large sample sizes and/or moderately reliable components are also discussed. An example based on lognormal data is given to illustrate the theory.

1. INTRODUCTION.

In life testing, the two-parameter lognormal density

$$f_T(t; \mu, \sigma) = \left[(2\pi)^{1/2} \sigma t \right]^{-1} \exp \left\{ - \left[(\ln t - \mu) / \sigma \right]^2 / 2 \right\}, \quad t, \sigma > 0, \quad -\infty < \mu < +\infty \quad (1.1)$$

has been widely used to represent the distribution of certain failure-times.

The reliability function is

$$R(\tau; \mu, \sigma) = 1 - \Phi \left[(\ln \tau - \mu) / \sigma \right], \quad (1.2)$$

where τ is the "mission time", $\Phi(z) = \int_{-\infty}^z \phi(x) dx$, and $\phi(x) = (2\pi)^{-1/2} \exp(-x^2/2)$. Discussions of the lognormal distribution in reliability theory may be found in Epstein (1947) - (1948), Feinlieb (1960),

Goldthwaite (1961), and Nowick and Berry (1961). It is felt, at times, the lognormal is a more realistic failure-time distribution than the normal which assigns positive probability to negative failure-times. Furthermore, even if a normal distribution seems appropriate, it might be closely approximated by a lognormal distribution. Kotz (1973) and Klimko, et. al., (1975) have derived tests for normality vs. lognormality. The lognormal and the (two-parameter) Weibull are often competitors when a skewed distribution for a non-negative random variable is needed. For certain components which become "work-hardened" *only* after a certain period of operating time, the lognormal might be a more appropriate failure-time distribution than the Weibull. This is due to the fact that the lognormal hazard function initially increases with time to a maximum then decreases with increasing time. The Weibull hazard function (for a given set of parameters) can only increase, decrease, or remain constant with time. Dumonceaux and Antle (1973) provide a test for selecting between a lognormal model or a (two-parameter) Weibull model.

It is of considerable interest to construct point and interval estimates of (1.2) based on lognormal failure data. In most situations, by making a logarithmic transformation on the data, lognormal inferential problems may be reduced to corresponding normal inferential problems. Under the *classical* approach toward estimating (1.2) using transformed data from a complete sample of size n , the results are well known.

(a.) Point estimates of $R(\tau; \mu, \sigma)$. Because of the invariance property of maximum likelihood estimates, the maximum likelihood estimate of $R(\tau; \mu, \sigma)$ is given by $\hat{R}_{ML}(\tau; \mu, \sigma) =$

$$1 - \Phi \left[(\ln \tau - \hat{\mu}) / \hat{\sigma} \right], \quad \text{where } \hat{\mu} = n^{-1} \sum_{i=1}^n \ln t_i \quad \text{and}$$

$$\hat{\sigma} = \left[n^{-1} \sum_{i=1}^n (\ln t_i - \hat{\mu})^2 \right]^{1/2}. \quad \text{Lieberman and Resnikoff (1955)}$$

give a point estimate of $R(\tau; \mu, \sigma)$ which has the minimum variance unbiasedness property:

$$\hat{R}_{UMVU}(\tau; \mu, \sigma) = \begin{cases} 1 & , \quad \text{if } z(\hat{\mu}, \hat{\sigma}) < 0 \\ 1 - I_{z(p, q)} \left(\frac{n}{2} - 1, \frac{n}{2} - 1 \right) & , \quad \text{if } 0 \leq z(\hat{\mu}, \hat{\sigma}) \leq 1 \\ 0 & , \quad \text{if } z(\hat{\mu}, \hat{\sigma}) > 1 \end{cases} \quad (1.3)$$

where $\hat{\mu}$ and $\hat{\sigma}$ are the ML estimates, $I_z(p, q)$ is the incomplete beta function ratio, and

$$z(\hat{\mu}, \hat{\sigma}) = (1/2) \left[1 + (\ln \tau - \hat{\mu}) / (n - 1)^{1/2} \hat{\sigma} \right].$$

Zacks and Milton (1971) have compared the mean square errors of $\hat{R}_{ML}(\tau; \mu, \sigma)$ and $\hat{R}_{UMVU}(\tau; \mu, \sigma)$ and found $\hat{R}_{UMVU}(\tau; \mu, \sigma)$ to be more efficient than $\hat{R}_{ML}(\tau; \mu, \sigma)$ only if $0.3 < R(\tau; \mu, \sigma) \leq 0.7$.

- (b.) Lower confidence bound (LCB) for $R(\tau; \mu, \sigma)$. The problem of determining a LCB for $R(\tau; \mu, \sigma)$ may be solved by finding a lower tolerance limit on the normal distribution function. It can be shown [Zacks (1971), page 517] that a LCB for $R(\tau; \mu, \sigma)$ with confidence level $1 - \alpha$ is

$$\underline{R} = 1 - \Phi(-\lambda^* / \sqrt{n}) \quad , \quad (1.4)$$

where λ^* is the value of the noncentrality parameter which satisfies the equation $\sqrt{n-1} a = t'_{1-\alpha; n-1, \lambda}$, where $a = (\hat{\mu} - \ln \tau) / \hat{\sigma}$, and $t'_{1-\alpha; n-1, \lambda}$ is the $100(1 - \alpha)$ th percentile of a noncentral t -distribution with $n - 1$ degrees of freedom and parameter of noncentrality λ . For t', n, α given, the value of λ may be determined from tables in Owen (1962), pp. 113-116.

In this paper we shall consider the structural approach to statistical inference on the reliability function $R(\tau; \mu, \sigma)$. The essentials of the structural model will be given in the next section. Based on the structural density for the two-parameter lognormal distribution, the structural density and distribution function for the reliability function are derived. Consequently, expressions for structural point and interval estimates of the reliability function are developed. Graphical displays of the expressions are given for a few special cases of interest. Approximations for large sample sizes and/or moderately reliable components are also presented. A numerical example based on lognormal data is given to illustrate the theory.

2. THE STRUCTURAL MODEL

The theory of structural inference, as developed by Fraser (1968), is based on a group-theoretic approach using invariant Haar measures to Fisher's fiducial theory. Basically, the structural model may be described as follows. The model has two parts: an *error variable* E with a known distribution

$f(E)$ on the sample space $\mathcal{S}$ (an open set in Euclidean space R^N) of the response X ; and a *structural equation* $X = [\theta]E$ in which an observed response X in $\mathcal{S}$ is obtained by a transformation $[\theta]$ applied to a realized but unknown value from the error variable. Suppose the transformations $[\theta]$ are indexed by a parameter θ with values in a parameter space Ω and that the transformations $[\theta]$ are precisely the transformations of a group $G = \{g\}$ that is unitary in its application to the space $\mathcal{S}$, i.e., if $g_1x = g_2x$ for any x , then $g_1 = g_2$.

The transformations g in G carry a point X into the orbit of X : $GX = \{gX: g \in G\}$. Suppose that a reference point is chosen on each orbit and let $[X]$ represent the unique transformation in G that carries the reference point on the orbit GX into the point X . The reference point for the orbit through X can then be designated by $D(X) = [X]^{-1}X$. An orbit is indexed by $D(X)$, while $[X]$ gives position on an orbit [see Fraser (1968), pp. 49-51].

The observed response X together with the structural equation $X = [\theta]E$ and the above assumptions imply that E must lie in the orbit of X , but its position $[E]$ is not known. However, the conditional distribution of $[E]$, given $D(X)$, (called the error probability distribution) may be found. The structural distribution is obtained from the error probability distribution by the transformation $[\theta] = [X][E]^{-1}$.

The structural probability element for θ (we now write θ instead of $[\theta]$ since the group G is isomorphic to the parameter space Ω) on the space G is [Fraser (1968), page 64]

$$g(\theta|X)d\theta = k([X]^{-1}X) f(\theta^{-1}X) J_N(\theta^{-1}X) \frac{J_L^*(\theta^{-1}[X])}{J_L(\theta^{-1}[X])} \frac{d\theta}{J_L(\theta)}, \quad (2.1)$$

where k is a normalizing constant, $J_N(\theta^{-1}X) = J_N(\theta^{-1}:X)J_N(X)$

$= J_N(\theta^{-1}:X)J_N([X]:D)$ is the Jacobian $\left| \frac{\partial \theta^{-1}X}{\partial X} \right| \cdot \left| \frac{\partial [X]D}{\partial D} \right| = \left| \frac{\partial \theta^{-1}[X]D}{\partial D} \right|$,

$J_L(\theta^{-1}[X]) = J_L(\theta^{-1}[X]:i)$ is the Jacobian $\left| \frac{\partial \theta^{-1}[X]i}{\partial i} \right|$ associated with

a left invariant Haar measure and i is the identity in G , and

$J_L^*(\theta^{-1}[X])$ is the Jacobian $\left| \frac{\partial i \theta^{-1}[X]}{\partial i} \right|$ associated with a right invariant Haar measure.

Consider the group of affine transformations on R^n , $G = \{[\mu, \sigma]: -\infty < \mu < +\infty, 0 < \sigma < +\infty\}$, where $[\mu, \sigma]\underline{x} = [\mu, \sigma](x_1, \dots, x_n) = (\mu + \sigma x_1, \dots, \mu + \sigma x_n)$, i.e., the group of location-scale transformations. The product of two affine transformations is $[A, C][a, c] = [A + Ca, Cc]$, the identity is $[0, 1]$, and the inverse is $[a, c]^{-1} = [-c^{-1}a, c^{-1}]$. Under the observed response $X = \underline{x} = (x_1, x_2, \dots, x_n)$ and $[X] = [\bar{x}, s_x]$, where $\bar{x} = n^{-1} \sum_{i=1}^n x_i$ and $s_x = \left[n^{-1} \sum_{i=1}^n (x_i - \bar{x})^2 \right]^{1/2}$, we find

$$J_N(\theta^{-1}X) = \left| \frac{\partial [\mu, \sigma]^{-1}[\bar{x}, s_x]d}{\partial d} \right| = \left| \frac{\partial \left[-\frac{\mu}{\sigma} + \frac{\bar{x}}{\sigma} + \frac{s_x}{\sigma} \frac{d}{\sigma} \right]}{\partial d} \right| = \left(\frac{s_x}{\sigma} \right)^n,$$

$$J_L^*(\theta^{-1}[X]) = \left| \frac{\partial [a, c][\mu, \sigma]^{-1}[\bar{x}, s_x]}{\partial [a, c]} \right| = \left| \frac{\partial \left[a + c \left(-\frac{\mu}{\sigma} + \frac{\bar{x}}{\sigma} \right), c \frac{s_x}{\sigma} \right]}{\partial [a, c]} \right| = \frac{s_x}{\sigma},$$

$$J_L(\theta^{-1}[X]) = \left| \frac{\partial [\mu, \sigma]^{-1}[x, s_x][a, c]}{\partial [a, c]} \right| = \left| \frac{\partial \left[-\frac{\mu}{\sigma} + \frac{\bar{x}}{\sigma} + a \frac{s_x}{\sigma}, c \frac{s_x}{\sigma} \right]}{\partial [a, c]} \right| = \left(\frac{s_x}{\sigma} \right)^2,$$

$$J_L(\theta) = \left| \frac{\partial[\mu, \sigma][a, c]}{\partial[a, c]} \right| = \left| \frac{\partial[\mu + a\sigma, c\sigma]}{\partial[a, c]} \right| = \sigma^2 ,$$

where $\underline{d} = [\bar{x}, s_x]^{-1} \underline{x} = \left(\frac{x_1 - \bar{x}}{s_x}, \dots, \frac{x_n - \bar{x}}{s_x} \right)$.

We therefore find the structural probability element for μ and σ , given the observed value of $\underline{x}$, to be

$$g(\mu, \sigma | \underline{x}) d\mu d\sigma = k(\underline{d}) \prod_{i=1}^n f\left(\frac{x_i - \mu}{\sigma}\right) \left(\frac{s_x}{\sigma}\right)^n \frac{\frac{s_x}{\sigma}}{\left(\frac{s_x}{\sigma}\right)^2} \frac{d\mu d\sigma}{\sigma^2} \quad (2.2)$$

$$= K \prod_{i=1}^n \left[\frac{1}{\sigma} f\left(\frac{x_i - \mu}{\sigma}\right) \right] \frac{d\mu d\sigma}{\sigma} , \quad (2.3)$$

where K is a normalizing constant.

Equation (2.3) will be the basis for the derivations in this paper.

In essence, under the structural approach, the parameters of location-scale families of distributions are treated as *random variables*, their joint distribution being determined by information from the sample. Consequently, a classical model and a structural model are *fundamentally different* with regard to mathematical structure. In applications, they function differently, and different interpretations are given their results. Nevertheless, it is well known [Hora and Buehler (1966)] that within a certain class (the invariantly estimable functions) of interval estimation problems for families of distributions exhibiting location-scale structure, there is a "numerical equivalency" between the structural solution and classical (confidence interval) solution.

It should also be pointed out that a Bayesian interpretation of (2.3) may be given. The structural density of μ and σ is precisely the Bayes

posterior distribution when the prior measure is given by a right invariant Haar measure of $[\theta] = [\mu, \sigma]$, i.e., $\pi(\mu, \sigma) \propto 1/\sigma$. Similarities and dissimilarities of the classical, Bayes, and structural approaches to statistical inference are discussed in Fraser (1972). Applications of the theory of structural inference to other life-testing distributions may be found in Bury and Bernholtz [one-parameter exponential model (1971)] and Bury [two-parameter Weibull model (1972)].

3. STRUCTURAL DISTRIBUTION OF THE RELIABILITY FUNCTION

The lognormal density (1.1) does not have location-scale structure. However, the density of the logarithmic transformation of the measurement variable T (which is, of course, a normal density) does. Consequently, the structural joint density of the location and scale parameters μ and σ is

$$f(\mu, \sigma | a(\underline{x})) = c \prod_{i=1}^n \left[(2\pi)^{1/2} \sigma \right]^{-1} \exp \left\{ - \left[(x_i - \mu) / \sigma \right]^2 / 2 \right\} \cdot \sigma^{-1}, \quad (3.1)$$

$-\infty < \mu < +\infty$
 $\sigma > 0$

where $a(\underline{x})$ represents the sample information on which (3.1) is conditional, $x_i = \ln t_i$, and c is a normalizing constant. It should be noted that in the above density, μ and σ are random variables while the observations $\{x_i\}$ are given constants. The structural joint density of μ and σ , conditioned by $a(\underline{t})$, is

$$f(\mu, \sigma | a(\underline{t})) = c \sigma^{-n-1} (2\pi)^{-n/2} \exp \left[- \sum_{i=1}^n (\ln t_i - \mu)^2 / 2\sigma^2 \right], \quad (3.2)$$

$-\infty < \mu < +\infty$
 $\sigma > 0$

where

$$\begin{aligned}
 c^{-1} &= \int_0^\infty \int_{-\infty}^\infty f(\mu, \sigma | a(\underline{t})) d\mu d\sigma \\
 &= (2\pi)^{-n/2} \int_0^\infty \left(\int_{-\infty}^\infty \exp \left[-n \left(\mu - n^{-1} \sum_{i=1}^n \ln t_i \right)^2 / 2\sigma^2 \right] d\mu \right) \sigma^{-n-1} \exp \left(-n\sigma^2 / 2\sigma^2 \right) d\sigma \\
 &= \left[(2\pi)^{(n-1)/2} n^{1/2} \right]^{-1} \int_0^\infty \sigma^{-n} \exp \left(-n\sigma^2 / 2\sigma^2 \right) d\sigma \\
 &= \Gamma \left(\frac{n-1}{2} \right) \left[2n^{1/2} (\pi n\sigma^2)^{(n-1)/2} \right]^{-1},
 \end{aligned}$$

and

$$n\sigma^2 = \sum_{i=1}^n \left(\ln t_i - \hat{\mu} \right)^2.$$

Upon comparing (3.1) and (3.2), it might appear that we are not actually dealing with a lognormal distribution but, in fact, a normal distribution. In a general setting this is true. However, since this paper is concerned with applications in reliability theory, there is the problem of the normal distribution assigning positive probabilities to negative lifetimes. In a general setting, the range of μ in (3.1) is the real line. However in a reliability theory setting, it is *not* clear what range should be given μ (it would seem that the range of μ should be some subset of the positive real line). We therefore shall only use (3.1) to reach (3.2), and all results will be based on (3.2). Of course, *in a general setting*, the results obtained by using (3.2) can be carried over to a normal distribution upon replacing $\ln t_i$ by x_i .

Since μ and σ are considered as random variables, the reliability function (1.2) is also a random variable.

THEOREM 3.1.

The structural density function of the reliability function $R(\tau; \mu, \sigma)$ $= 1 - \Phi[(\ln \tau - \mu)/\sigma]$ is, for $n \geq 2$,
(parabolic cylinder function representation)

$$f_R\left(r|\tau, a(\underline{t})\right) = \frac{\sqrt{n} \pi \Gamma(n-1)}{2^{(n-5)/2} \Gamma\left(\frac{n-1}{2}\right) (1+a^2)^{(n-1)/2}} \quad (3.3)$$

$$\times \phi\left(\sqrt{(1-1/n)(1+1/a^2)} \omega\right) \phi\left(\omega/\sqrt{-2}\right) D_{-n+1}(\omega), \quad 0 < r < 1$$

or

(infinite series representation)

$$f_R\left(r|\tau, a(\underline{t})\right) = \frac{\sqrt{n} \sqrt{2\pi}}{\Gamma\left(\frac{n-1}{2}\right) (1+a^2)^{(n-1)/2}} \quad (3.4)$$

$$\times \phi\left(\sqrt{(1-1/n)(1+1/a^2)} \omega\right) \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma\left(\frac{j+n-1}{2}\right)}{j!} (\sqrt{2}\omega)^j, \quad 0 < r < 1$$

where $\omega = \sqrt{n} \Phi^{-1}(1-r)a/\sqrt{1+a^2}$, $D_{-n+1}(\omega)$ is the parabolic cylinder

function, and $a = (\hat{\mu} - \ln \tau)/\hat{\sigma}$.

PROOF.

The structural density of R is a derived distribution from (3.2).

We have

$$\begin{aligned}
 f_R\left(r|\tau, a(\underline{t})\right) &= \int_0^\infty f\left(\ln \tau - \sigma \Phi^{-1}(1-r), \sigma|a(\underline{t})\right) \cdot \left| \frac{d\left[\ln \tau - \sigma \Phi^{-1}(1-r)\right]}{dr} \right| d\sigma \\
 &= \frac{\sqrt{n} \sqrt{2\pi} \phi\left(\sqrt{(1-1/n)(1+1/a^2)}\omega\right)}{2^{(n-3)/2} \Gamma\left(\frac{n-1}{2}\right) (1+a^2)^{(n-1)/2}} \\
 &\quad \times \int_0^\infty y^{n-2} \exp(-y^2/2 - \omega y) dy, \quad (3.5)
 \end{aligned}$$

where ω and a are as defined in the theorem. We have also used the

result that $(d/dr)\Phi^{-1}(1-r) = -\left\{\phi\left[\Phi^{-1}(1-r)\right]\right\}^{-1}$.

The integral on the right-hand side of (3.5) may be written as $\sqrt{2\pi} \Gamma(n-1) \phi(\omega/\sqrt{2}) D_{-n+1}(\omega)$, for $n \geq 2$, where $D_\gamma(x)$ is the parabolic cylinder function [Erdelyi, et. al., II (1953), page 119].

Lebedev (1965, page 288) has shown, through the use of Hermite functions, that a parabolic cylinder function may be represented by the series

$$D_{-n+1}(x) = \frac{2^{(n-3)/2} \exp(-x^2/4)}{\Gamma(n-1)} \sum_{j=0}^{\infty} \frac{(-1)^j \Gamma\left(\frac{j+n-1}{2}\right)}{j!} (\sqrt{2}x)^j$$

thus completing the proof.

To find the structural distribution function of R , we shall use the

series representation of the structural density function of R . When
 $0 \leq r \leq 1/2$,

$$F_R\left(r|\tau, a(\underline{t})\right) = \frac{\sqrt{n}}{\Gamma\left(\frac{n-1}{2}\right)(1+a^2)^{(n-1)/2}} \sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{j+n-1}{2}\right)}{j!} \left(\frac{-\sqrt{2n}a}{\sqrt{1+a^2}}\right)^j \mathfrak{E}_j^{(0,r)},$$

where

$$\mathfrak{E}_j^{(a,b)} = \int_a^b \left[\Phi^{-1}(1-t)\right]^j \exp\left\{-(n-1)\left[\Phi^{-1}(1-t)\right]^2/2\right\} dt.$$

Making the change of variable $y = n\left[\Phi^{-1}(1-t)\right]^2$, we have

$$\mathfrak{E}_j^{(0,r)} = \frac{2^{(j-2)/2} \Gamma\left(\frac{j+1}{2}\right)}{\sqrt{\pi} n^{(j+1)/2}} \left\{ 1 - \chi_{j+1}^2 \left[n \left[\Phi^{-1}(1-r)\right]^2 \right] \right\},$$

where $\chi_k^2(x)$ is the chi-square distribution function with k degrees of freedom.

Upon using the duplication formula

$$\Gamma(2x) = (2\pi)^{-1/2} 2^{2x-1/2} \Gamma(x) \Gamma(x+1/2),$$

we have

$$F_R\left(r|\tau, a(\underline{t})\right) = \frac{1}{2\Gamma\left(\frac{n-1}{2}\right)(1+a^2)^{(n-1)/2}} \times \quad (3.6)$$

$$\sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{j+n-1}{2}\right)}{\Gamma\left(\frac{j}{2}+1\right)} \left(\frac{-a}{\sqrt{1+a^2}}\right)^j \left\{ 1 - \chi_{j+1}^2 \left[n \left[\Phi^{-1}(1-r)\right]^2 \right] \right\}.$$

When $1/2 \leq r \leq 1$,

$$F_R\left(r|\tau, a(\underline{t})\right) = F_R\left(1/2|\tau, a(\underline{t})\right) + \frac{\sqrt{n}}{\Gamma\left(\frac{n-1}{2}\right)(1+a^2)^{(n-1)/2}} \sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{j+n-1}{2}\right)}{j!} \left(\frac{-\sqrt{2n}a}{\sqrt{1+a^2}}\right)^j \mathfrak{E}_j(1/2, r) .$$

Making the change of variable $y = n\left[\Phi^{-1}(1-t)\right]^2$, we find

$$\mathfrak{E}_j(1/2, r) = \frac{(-1)^j 2^{j/2} \Gamma\left(\frac{j+1}{2}\right)}{\sqrt{\pi} n^{(j+1)/2}} \chi_{j+1}^2 \left(n \left[\Phi^{-1}(1-r) \right]^2 \right) ,$$

so that

$$F_R\left(r|\tau, a(\underline{t})\right) = F_R\left(1/2|\tau, a(\underline{t})\right) + \frac{1}{2\Gamma\left(\frac{n-1}{2}\right)(1+a^2)^{(n-1)/2}} \sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{j+n-1}{2}\right)}{\Gamma\left(\frac{j}{2}+1\right)} \left(\frac{a}{\sqrt{1+a^2}}\right)^j \chi_{j+1}^2 \left(n \left[\Phi^{-1}(1-r) \right]^2 \right) . \quad (3.7)$$

The value of the distribution function at $r = 1/2$ will be of special interest. From (3.6),

$$\begin{aligned} F_R\left(1/2|\tau, a(\underline{t})\right) &= \frac{1}{2\Gamma\left(\frac{n-1}{2}\right)(1+a^2)^{(n-1)/2}} \sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{j+n-1}{2}\right)}{\Gamma\left(\frac{j}{2}+1\right)} \left(\frac{-a}{\sqrt{1+a^2}}\right)^j \\ &= \frac{1}{2\Gamma\left(\frac{n-1}{2}\right)(1+a^2)^{(n-1)/2}} \left\{ \sum_{j=0}^{\infty} \frac{\Gamma\left(j+\frac{n-1}{2}\right)}{\Gamma(j+1)} \left(\frac{a^2}{1+a^2}\right)^j \right. \\ &\quad \left. - \frac{a}{\sqrt{1+a^2}} \sum_{j=0}^{\infty} \frac{\Gamma\left(j+\frac{n}{2}\right)}{\Gamma\left(j+\frac{3}{2}\right)} \left(\frac{a^2}{1+a^2}\right)^j \right\} . \quad (3.8) \end{aligned}$$

Each series on the right-hand side of (3.8) may be written in terms of a hypergeometric function

$${}_2F_1(\alpha, \beta; \gamma; z) = \sum_{k=0}^{\infty} \frac{(\alpha)_k (\beta)_k}{(\gamma)_k k!} z^k, \quad |z| < 1$$

where $(\alpha)_k = \Gamma(\alpha + k)/\Gamma(\alpha)$. The first series is [Erdélyi, et. al., I (1953), page 101]

$$\begin{aligned} \sum_{j=0}^{\infty} \frac{\Gamma\left(j + \frac{n-1}{2}\right)}{\Gamma(j+1)} \left[\frac{a^2}{1+a^2} \right]^j &= \Gamma\left(\frac{n-1}{2}\right) {}_2F_1\left((n-1)/2, \beta; \beta; a^2/(1+a^2)\right) \\ &= \Gamma\left(\frac{n-1}{2}\right) \left[1 - a^2/(1+a^2)\right]^{-(n-1)/2} = \Gamma\left(\frac{n-1}{2}\right) (1+a^2)^{(n-1)/2}. \end{aligned} \quad (3.9)$$

In order to evaluate the second series, we first establish a relationship between the hypergeometric function and an F -distribution function. It is well known that

$$1 - F_{\gamma_1, \gamma_2}(y) = I_x\left(\gamma_2/2, \gamma_1/2\right), \quad (3.10)$$

where $F_{\gamma_1, \gamma_2}(y)$ is the F -distribution function with γ_1 and γ_2 degrees of freedom, $x = \gamma_2/(\gamma_2 + \gamma_1 y)$, and $I_x(\alpha, \beta)$ is the incomplete beta function ratio. However

$$\begin{aligned} I_x(\gamma_2/2, \gamma_1/2) &= \frac{(\gamma_2/2)^{-1} x^{\gamma_2/2}}{B(\gamma_1/2, \gamma_2/2)} {}_2F_1(\gamma_2/2, 1 - \gamma_1/2; 1 + \gamma_2/2; x) \\ &= \frac{(\gamma_2/2)^{-1} x^{\gamma_2/2} (1-x)^{\gamma_1/2}}{B(\gamma_1/2, \gamma_2/2)} {}_2F_1\left(1, (\gamma_1 + \gamma_2)/2; 1 + \gamma_2/2; x\right) \end{aligned} \quad (3.11)$$

by a linear transformation of the hypergeometric function [Erdélyi, et. al., I (1953), page 64]. Combining (3.10) and (3.11) yields the desired result

$${}_2F_1\left(1, (\gamma_1 + \gamma_2)/2; 1 + \gamma_2/2; x\right) = \frac{B(\gamma_1/2, \gamma_2/2)(\gamma_2/2)}{x\gamma_2^{1/2}(1-x)\gamma_1^{1/2}} \left[1 - F_{\gamma_1, \gamma_2}(y)\right], \quad (3.12)$$

where $y = \gamma_2(1-x)/\gamma_1 x$, $x \neq 0$. The second series on the right-hand side of (3.8) is now

$$\begin{aligned} \sum_{j=0}^{\infty} \frac{\Gamma\left(j + \frac{n}{2}\right)}{\Gamma\left(j + \frac{3}{2}\right)} \left(\frac{a^2}{1+a^2}\right)^j &= \frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{3}{2}\right)} {}_2F_1\left(1, n/2; 3/2; a^2/(1+a^2)\right) \\ &= \begin{cases} \frac{\Gamma\left(\frac{n-1}{2}\right)(1+a^2)^{n/2}}{|a|} F_{1, n-1}\left[(n-1)a^2\right], & a \neq 0 \\ \Gamma(n/2)/\Gamma(3/2), & a = 0. \end{cases} \end{aligned} \quad (3.13)$$

Finally, upon substituting (3.9) and (3.13) into (3.8), we have the value of the distribution function at $r = 1/2$,

$$F_R\left(1/2 | \tau, a(\underline{t})\right) = (1/2) \left\{ 1 - \operatorname{sgn}(a) F_{1, n-1}\left[(n-1)a^2\right] \right\}, \quad (3.14)$$

where $\operatorname{sgn}(a) = 1$, $a > 0$; -1 , $a < 0$; $= 0$, $a = 0$.

By combining (3.6), (3.7), and (3.14), we may now write the following

THEOREM 3.2.

The structural distribution function of the reliability function

$R(\tau; \mu, \sigma)$ is, for $0 \leq r \leq 1$,

$$\begin{aligned} F_R\left(r | \tau, a(\underline{t})\right) &= (1/2) \left\{ 1 - \operatorname{sgn}(a) F_{1, n-1}\left[(n-1)a^2\right] \right. \\ &\quad \left. + \frac{\operatorname{sgn}(r - 1/2)}{(1+a^2)^{(n-1)/2}} \sum_{j=0}^{\infty} \frac{\Gamma\left(j + \frac{n-1}{2}\right)}{\Gamma\left(\frac{j}{2} + 1\right)} \frac{\Gamma\left(\frac{n-1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)} \left[\frac{\operatorname{sgn}(r - 1/2)a}{\sqrt{1+a^2}}\right]^j \chi_{j+1}^2 \left(n \left[\Phi^{-1}(1-r) \right]^2 \right) \right\} \end{aligned} \quad (3.15)$$

Since the value of r in (3.15) is unknown, the following lemma will be useful. The proof is straightforward.

LEMMA.

Let a^* be the solution for a in the equation. $F_R\left[1/2|\tau, a(\underline{t})\right] = \alpha (\alpha \leq 1/2)$. For fixed α and n , $\text{sgn}(r - 1/2) = 1$, i.e., $1/2 < r \leq 1$, iff $a > a^* = \left[(n-1)^{-1} F_{1,n-1}^{-1}(1-2\alpha)\right]^{1/2}$.

4. STRUCTURAL INFERENCE ON THE RELIABILITY FUNCTION

We shall now use the distributions of Section 3 to obtain point and interval estimates of the reliability function $R(\tau; \mu, \sigma)$.

(i.) *Point estimates.* Measures of central tendency of the structural density would seem natural choices for point estimates. Furthermore, because of the Bayesian interpretation discussed in Section 2, the mean, median, and mode of the structural density correspond to the Bayesian point estimate of $R(\tau; \mu, \sigma)$ using a squared error, absolute error, and simple loss function, respectively, and a prior measure given by $\pi(\mu, \sigma) \propto 1/\sigma$ (a right invariant Haar measure).

THEOREM 4.1.

Based on the structural distribution of the reliability function $R(\tau; \mu, \sigma)$, the mean of the distribution of $R(\tau; \mu, \sigma)$ is

$$E\left[R(\tau; \mu, \sigma)\right] = (1/2) \left\{ 1 + \frac{a}{(1+a^2)^{n/2}} \times \sum_{j=0}^{\infty} \frac{\Gamma(j+n/2)}{\Gamma\left[j+3/2\right] \Gamma\left[\frac{n-1}{2}\right]} \left(\frac{a^2}{1+a^2}\right)^j F_{1,2(j+1)}\left[2(j+1)/n\right] \right\}; \quad (4.1)$$

the *median* of the distribution of $R(\tau; \mu, \sigma)$ is the value of r which satisfies the equation

$$r = 1 - \Phi \left[\frac{-\operatorname{sgn}(a)}{\sqrt{n}} \Phi^{-1} \left(\frac{1+A}{2} \right) \right], \quad (4.2)$$

where

$$A = (1 + a^2)^{(n-1)/2} F_{1, n-1} \left[(n-1)a^2 \right] - \sum_{j=1}^{\infty} \frac{\Gamma \left(\frac{j+n-1}{2} \right)}{\Gamma(j/2+1) \Gamma \left(\frac{n-1}{2} \right)} \left(\frac{\operatorname{sgn}(a)a}{\sqrt{1+a^2}} \right)^j \chi_{j+1}^2 \left(n \left[\Phi^{-1}(1-r) \right]^2 \right);$$

the *mode* of the distribution of $R(\tau; \mu, \sigma)$, r_{mode} , is given by

$$r_{\text{mode}} = 1 - \Phi \left[\frac{\sqrt{1+a^2}}{\sqrt{n} a} \xi \right], \quad (4.3)$$

where ξ is the solution to the equation

$$D_{-n+2}(\xi) = \left[n(1+a^2)^{-1} - 1 \right] D_{-n}(\xi). \quad (4.4)$$

PROOF.

Since $R(\tau; \mu, \sigma)$ is a positive random variable,

$$\begin{aligned} E[R(\tau; \mu, \sigma)] &= \int_0^1 \left[1 - F_R(r | \tau, a(\underline{t})) \right] dr \\ &= (1/2) \left\{ 1 + \operatorname{sgn}(a) F_{1, n-1} \left[(n-1)a^2 \right] \right\} \\ &\quad - \frac{a}{(1+a^2)^{n/2}} \sum_{j=0}^{\infty} \frac{\Gamma(j+n/2)}{\Gamma(j+3/2) \Gamma \left(\frac{n-1}{2} \right)} \left(\frac{a^2}{1+a^2} \right)^j \int_0^{1/2} \chi_{2j+2}^2 \left(n \left[\Phi^{-1}(1-r) \right]^2 \right) dr. \end{aligned}$$

Using the relationship between the chi-squared distribution function and the incomplete gamma function $\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1} e^{-t} dt$, i.e., $\chi_{2j+2}^2(y) =$

$1 - \Gamma\left\{j+1, n\left[\Phi^{-1}(1-r)\right]^2/2\right\} / \Gamma(j+1)$, we have

$$\begin{aligned} & \int_0^{1/2} \chi_{2j+2}^2\left(n\left[\Phi^{-1}(1-r)\right]^2/2\right) dr \\ &= 1/2 - \frac{1}{\Gamma(j+1)} \int_0^{1/2} \Gamma\left\{j+1, n\left[\Phi^{-1}(1-r)\right]^2/2\right\} dr \\ &= 1/2 - \frac{1}{2\sqrt{n\pi} \Gamma(j+1)} \int_0^\infty y^{-1/2} \exp(-y/n) \Gamma(j+1, y) dy \\ &= 1/2 - \frac{\Gamma(j+3/2)}{\sqrt{n\pi} \Gamma(j+1) (1+1/n)^{j+3/2}} {}_2F_1\left(1, j+3/2; 3/2; 1/(n+1)\right) \\ &= 1/2 \left\{ 1 - F_{1,2(j+1)}\left[2(j+1)/n\right] \right\} \end{aligned}$$

using (3.12). The result follows from (3.13).

The median of the distribution of $R(\tau; \mu, \sigma)$ is the value of r which satisfies the equation $F_R\left[r|\tau, a(\underline{t})\right] = 1/2$. The proof of (4.2) will be given in a slightly more general version toward the end of this section.

The mode of the distribution of $R(\tau; \mu, \sigma)$ is the value of r , say r_{mode} , which maximizes the structural density. We first differentiate (3.3) with respect to r using the following expression for the parabolic cylinder function from Erdélyi, et. al., II (1953), page 119

$$(d/dx) \left[\exp(x^2/4) D_\gamma(x) \right] = \gamma \exp(x^2/4) D_{\gamma-1}(x) .$$

Equating the first derivative to zero yields

$$\begin{aligned} & \phi^{-1}(1-r) D_{-n+1} \left(\sqrt{n} \phi^{-1}(1-r)a/\sqrt{1+a^2} \right) \\ & + (\sqrt{n} a/\sqrt{1+a^2}) D_{-n} \left(\sqrt{n} \phi^{-1}(1-r)a/\sqrt{1+a^2} \right) = 0 \end{aligned}$$

or equivalently,

$$\omega D_{-n+1}(\omega) + \left[na^2/(1+a^2) \right] D_{-n}(\omega) = 0 ,$$

where $\omega = \sqrt{n} \phi^{-1}(1-r)a/\sqrt{1+a^2}$. By using the recurrence relation

$$D_{\gamma+1}(\omega) - \omega D_{\gamma}(\omega) + \gamma D_{\gamma-1}(\omega) = 0 ,$$

the proof is completed.

A few remarks concerning Theorem 4.1 are in order. In most cases, only a few terms are needed to evaluate the sum in equation (4.1). For the convenience of the reader, Figure 1 graphically displays na vs. $E[R(\tau; \mu, \sigma)]$ for $n = 5(5)25$. Equation (4.2) can be solved iteratively by using the value of $E[R(\tau; \mu, \sigma)]$ as a starting value. Again, in most instances, only a few terms are needed to evaluate the sum. Equation (4.4) can be readily solved using tables of the parabolic cylinder function given in Abramowitz and Stegun (1965), pp. 702-711.

(ii.) *Interval estimates.* Since the reliability function $R(\tau; \mu, \sigma)$ is a random variable, we wish to determine $\underline{r}(1-\alpha)$ where $P[\underline{r}(1-\alpha) < R(\tau; \mu, \sigma)] = 1-\alpha$, i.e., $F_R(\underline{r}(1-\alpha) | \tau, a(t)) = \alpha$, for some pre-assigned α . We shall call $\underline{r}(1-\alpha)$ the $(1-\alpha)$ -structural lower bound for $R(\tau; \mu, \sigma)$. It is, in this case, "numerically equivalent" to the $(1-\alpha)$ -Bayes lower confidence bound under the prior right invariant Haar

measure and to the $(1-\alpha)$ -classical (in the sense of Neyman) lower confidence bound for $R(\tau; \mu, \sigma)$ given by (1.4).

THEOREM 4.2.

Based on the structural distribution of the reliability function $R(\tau; \mu, \sigma)$, the $(1-\alpha)$ -structural lower bound for $R(\tau; \mu, \sigma)$ is the solution to the equation

$$r = 1 - \Phi \left\{ -\text{sgn}(r - 1/2) \Phi^{-1} \left[(1 + A^*)/2 \right] / \sqrt{n} \right\}, \quad (4.5)$$

where

$$A^* = (1 + a^2)^{(n-1)/2} \text{sgn}(r - 1/2) \left\{ \text{sgn}(a) F_{1, n-1}[(n-1)a^2] - (1 - 2\alpha) \right\} - \sum_{j=1}^{\infty} \frac{\Gamma\left(\frac{j+n-1}{2}\right)}{\Gamma\left(\frac{j}{2}+1\right) \Gamma\left(\frac{n-1}{2}\right)} \left(\frac{\text{sgn}(r - 1/2)a}{\sqrt{1+a^2}} \right)^j \chi_{j+1}^2 \left[n \left[\Phi^{-1}(1-r) \right]^2 \right]. \quad (4.6)$$

PROOF.

The equation $F_R[r|\tau, a(\underline{t})] = \alpha$ may be written

$$\begin{aligned} & \text{sgn}(a) F_{1, n-1}[(n-1)a^2] - (1-2\alpha) \\ &= \frac{\text{sgn}(r - 1/2)}{(1 + a^2)^{(n-1)/2}} \sum_{j=0}^{\infty} \frac{\Gamma\left(\frac{j+n-1}{2}\right)}{\Gamma\left(\frac{j}{2}+1\right) \Gamma\left(\frac{n-1}{2}\right)} \left(\frac{\text{sgn}(r - 1/2)a}{\sqrt{1+a^2}} \right)^j \chi_{j+1}^2 \left[n \left[\Phi^{-1}(1-r) \right]^2 \right] \end{aligned}$$

or equivalently,

$$\chi_1^2 \left[n \left[\Phi^{-1}(1-r) \right]^2 \right] = A^* > 0, \quad (4.7)$$

where A^* is given by (4.6). But it can be shown that if $\chi_1^2(y) = A^*$, then $\sqrt{y} = \Phi^{-1}[(1 + A^*)/2]$. Consequently, (4.7) becomes $|\Phi^{-1}(1-r)| = \Phi^{-1}[(1 + A^*)/2]/\sqrt{n}$ so that $\Phi^{-1}(1-r) = -\text{sgn}(r - 1/2)\Phi^{-1}[(1+A^*)/2]/\sqrt{n}$. Equation (4.5) now follows.

Given the data, i.e., the values of n and α , the lemma of Section 3 indicates the value of $\text{sgn}(r - 1/2)$ for specified α . Equation (4.5) can then be solved iteratively using any one of the three point estimates of Theorem 4.1 as a starting value. In most cases, only the first few terms in the series in (4.6) need be computed. Figures 2 and 3 graphically display α v.s. $|r(1-\alpha)|$ for $\alpha = 0.10, 0.05$ and $n = 5(5)25$.

5. APPROXIMATIONS

The expressions for the structural point and interval estimates given in the previous section may be evaluated with the aid of existing tables and/or a computer. It would, however, be desirable to find approximations to these estimates which are both accurate and easier to calculate.

The starting point will be the approximation of the structural density of $R(\tau; \mu, \sigma)$ given by (3.3) and, in particular, the parabolic cylinder function $D_{-n+1}(\omega)$, where $\omega = \sqrt{n} \Phi^{-1}(1-r)\alpha/\sqrt{1+\alpha^2}$. For the moment, we shall focus our attention on a large-sample approximation. For convenience, we shall use the alternate (and more contemporary) expression

$$U(n - 3/2, \omega) = D_{-n+1}(\omega). \quad (5.1)$$

Although there exist several asymptotic expansions for $U(n - 3/2, \omega)$, most do not allow for the doubly asymptotic nature of the parabolic cylinder function being considered (unbounded n and ω). Fortunately, a general

theory for the asymptotic solution of second-order linear differential equations has been developed by Olver (1959) which is applicable in this case. Specifically, for large n ,

$$U(n - 3/2, \sqrt{2(2n-3)} t) \approx \bar{g}(n) (t^2 + 1)^{-1/4} \exp[-(n - 3/2)\bar{\xi}(t)], \quad (5.2)$$

provided $-1 + \delta < t < 1 - \delta$, where $\bar{g}(n)$ is a function of n and $\bar{\xi}(t) = t(t^2 + 1)^{1/2} + \ln[t + (t^2 + 1)^{1/2}]$. However,

$$t(t^2 + 1)^{1/2} = t + \frac{1}{2} t^3 - \frac{1 \cdot 1}{2 \cdot 4} t^5 + \dots, \quad |t| < 1$$

and

$$\ln[t + (t^2 + 1)^{1/2}] = t - \frac{1}{2 \cdot 3} t^3 + \frac{1 \cdot 3}{2 \cdot 4 \cdot 5} t^5 - \dots, \quad |t| < 1$$

so that $\bar{\xi}(t) \approx 2t$ provided $-1 + \delta < t < 1 - \delta$.

Recall that in (5.1), $\omega = \sqrt{n} \Phi^{-1}(1-r)a/\sqrt{1+a^2}$. Consequently, in (5.2), $t = (1/2)[2n/(2n-3)]^{1/2} \Phi^{-1}(1-r)a/\sqrt{1+a^2}$. It follows that $-1 + \delta < t < 1 - \delta$ for large n , all values of a , and all but extreme values (i.e., $r < 0.01$ or $r > 0.99$) of r .

We shall also use the expansion

$$(1 + t^2)^{-1/4} = 1 - \frac{1}{4} t^2 + \frac{1 \cdot 5}{4 \cdot 4} \frac{t^4}{2} - \dots, \quad |t| < 1.$$

Therefore, the large sample structural density of $R(\tau; \mu, \sigma)$ is taken to be

$$\begin{aligned} \tilde{f}_R(r|\tau, a(\underline{t})) = k & \left\{ 1 - (1/16)[2n/(2n-3)][\Phi^{-1}(1-r)]^2 a^2/(1+a^2) \right\} \\ & \times \exp \left\{ -(1/2) \left[\frac{n(2+a^2)}{2(1+a^2)} - 1 \right] [\Phi^{-1}(1-r)]^2 \right. \\ & \left. - (1/2)[2n(2n-3)]^{1/2} \Phi^{-1}(1-r)a/\sqrt{1+a^2} \right\}, \quad 0 < r < 1 \end{aligned} \quad (5.3)$$

where k is a normalizing constant.

The following lemma will be needed in evaluating k as well as the distribution function of (5.3).

LEMMA.

$$\int_0^{c_3} \left[\Phi^{-1}(1-t) \right]^{2j} \exp \left\{ -(1/2) \left\{ c_1 \left[\Phi^{-1}(1-t) \right]^2 + 2c_2 \Phi^{-1}(1-t) \right\} \right\} dt$$

$$= (1+c_1)^{-1/2} \exp \left[c_2^2 / 2(1+c_1) \right] \times \quad (5.4)$$

$$\left\{ 1 - \Phi \left[(1+c_1)^{1/2} \Phi^{-1}(1-c_3) + c_2(1+c_1)^{-1/2} \right] \right\} \mu'_{2j}$$

for $j = 0, 1, 2, \dots$ and $c_1, c_3 > 0$,

where μ'_{2j} is the $2j^{\text{th}}$ moment of a right-truncated normal distribution with mean $c_2/(1+c_1)$ and variance $1/(1+c_1)$ with upper truncation point $-\Phi^{-1}(1-c_3)$.

PROOF.

In the integral on the left-hand side of (5.4), we make the transformation $y = -\Phi^{-1}(1-t)$, $dt = \phi(-y)dy$. Then the integral may be written

$$(2\pi)^{-1/2} \int_{-\infty}^{-\Phi^{-1}(1-c_3)} y^{2j} \exp \left\{ -(1/2) \left[(c_1 + 1)y^2 - 2c_2 y \right] \right\} dy$$

$$= (1+c_1)^{-1/2} \exp \left[c_2^2 / 2(1+c_1) \right] \times$$

$$\int_{-\infty}^{-\Phi^{-1}(1-c_3)} y^{2j} \frac{1}{(1+c_1)^{-1/2}} \phi \left[\frac{y - c_2(1+c_1)^{-1}}{(1+c_1)^{-1/2}} \right] dy$$

$$= (1+c_1)^{-1/2} \exp \left[c_2^2 / 2(1+c_1) \right] \Phi \left[\frac{-\Phi^{-1}(1-c_3) - c_2(1+c_1)^{-1}}{(1+c_1)^{-1/2}} \right] \times$$

$$\int_{-\infty}^{-\Phi^{-1}(1-c_3)} \frac{y^{2j} \frac{1}{(1+c_1)^{-1/2}} \Phi \left[\frac{y - c_2(1+c_1)^{-1}}{(1+c_1)^{-1/2}} \right]}{\Phi \left[\frac{-\Phi^{-1}(1-c_3) - c_2(1+c_1)^{-1}}{(1+c_1)^{-1/2}} \right]} dy.$$

The result follows by using the relationship $\Phi(-x) = 1 - \Phi(x)$.

The value of the normalizing constant k is now easily found by using (5.4) with $c_1 = [n(2 + a^2)/2(1 + a^2)] - 1$, $c_2 = (1/2)\sqrt{2n(2n-3)} \times a/\sqrt{1+a^2}$, $c_3 = 1$, and noting that μ'_{2j} is the $2j^{\text{th}}$ moment of an (un-truncated) normal distribution with mean $\lambda_1 = \sqrt{2(2n-3)} a\sqrt{1+a^2}/\sqrt{n} \times (2+a^2)$ and standard deviation $\lambda_2 = \sqrt{2(1+a^2)/n(2+a^2)}$. Upon simplification,

$$k^{-1} = (1+c_1)^{-1/2} \exp \left[c_2^2/2(1+c_1) \right] \left[1 - \frac{n a^2 (\lambda_1^2 + \lambda_2^2)}{8(2n-3)(1+a^2)} \right]. \quad (5.5)$$

The distribution function corresponding to (5.3) is, with the aid of (5.4),

$$\begin{aligned} \tilde{F}_R \left(r | \tau, a(\underline{t}) \right) &= \int_0^r \tilde{f}_R \left(t | \tau, a(\underline{t}) \right) dt \\ &= k(1+c_1)^{-1/2} \exp \left[c_2^2/2(1+c_1) \right] \left[1 - \frac{n a^2}{8(2n-3)(1+a^2)} \mu_2^{*'} \right] \\ &\quad \times \left\{ 1 - \Phi \left[\frac{\Phi^{-1}(1-r) + \lambda_1}{\lambda_2} \right] \right\}, \end{aligned} \quad (5.6)$$

where $\mu_2^{*'}$ is the second moment of a right-truncated normal distribution with mean $\lambda_1 = \sqrt{2(2n-3)} a\sqrt{1+a^2}/\sqrt{n} (2+a^2)$ and standard deviation

$\lambda_2 = \sqrt{2(1 + a^2)/n(2 + a^2)}$ and upper truncation point $-\Phi^{-1}(1 - r)$. From Johnson and Kotz (1970), page 83,

$$\mu_2^{*'} = \lambda_1^2 + \lambda_2^2 + \left[\frac{\Phi^{-1}(1 - r) - \lambda_1}{\lambda_2} \right] \lambda_2^2 \frac{\phi(u)}{1 - \Phi(u)}, \quad (5.7)$$

where $u = [\Phi^{-1}(1 - r) + \lambda_1] / \lambda_2$.

By combining (5.5), (5.6), and (5.7), we obtain the following

THEOREM 5.1.

The large-sample structural distribution function of $R(\tau; \mu, \sigma)$ is

$$\tilde{F}_R \left(r | \tau, a(\underline{t}) \right) = 1 - \Phi(u) - \frac{(u - 2\lambda_1/\lambda_2) \lambda_2^2 \phi(u)}{\left[8(2n - 3)(1 + a^2)/na^2 \right] - (\lambda_1^2 + \lambda_2^2)}, \quad (5.8)$$

where

$$u = [\Phi^{-1}(1 - r) + \lambda_1] / \lambda_2,$$

$$\lambda_1 = \sqrt{2(2n - 3)} a \sqrt{1 + a^2} / \sqrt{n(2 + a^2)},$$

$$\lambda_2 = \sqrt{2(1 + a^2)/n(2 + a^2)}.$$

THEOREM 5.2.

(i.) A large-sample point estimate of $R(\tau; \mu, \sigma)$ is

$$\tilde{E} \left[R(\tau; \mu, \sigma) \right] = \Phi \left(\frac{\lambda_1}{\sqrt{1 + \lambda_2^2}} \right) - \theta \frac{\lambda_1(2 + \lambda_2^2)}{\lambda_2(1 + \lambda_2^2)} \Phi \left(\frac{\lambda_1}{\sqrt{1 + \lambda_2^2}} \right), \quad (5.9)$$

where $\theta = \frac{\lambda_2^2}{\left[8(2n - 3)(1 + a^2)/na^2 \right] - (\lambda_1^2 + \lambda_2^2)};$

(ii.) A large-sample $(1 - \alpha)$ -structural lower bound for $R(\tau; \mu, \sigma)$ is given by

$$\tilde{r}(1 - \alpha) = 1 - \Phi(\lambda_2 \tilde{\Delta} - \lambda_1) , \quad (5.10)$$

where $\tilde{\Delta}$ is the solution to the equation

$$\Delta = \Phi^{-1} \left[1 - \alpha - \theta(\Delta - 2\lambda_1/\lambda_2) \phi(\Delta) \right] .$$

Although (5.3) was derived under the assumption of a large sample size n , it may also be used for *any* n provided $|a|$ is sufficiently small, say $|a| \leq 1/2$. To see this is the case, we first note that $1 \gg \left[\Phi^{-1}(1 - r) \right]^2 a^2 / (1 + a^2)$ for small $|a|$. Consequently, by an expansion of the parabolic cylinder function given in Abramowitz and Stegun (1965), page 689, we have

$$U(n - 3/2, \omega) \approx \bar{h}(n) \exp(-\sqrt{n - 3/2} \omega) ,$$

where $\omega = \sqrt{n} \Phi^{-1}(1 - r) a / \sqrt{1 + a^2}$ and $\bar{h}(n)$ is a function of n . Based on this result, an approximation to the structural density of $R(\tau; \mu, \sigma)$ is given by (5.3) except that the term $(1/16)[2n/(2n - 3)] \left[\Phi^{-1}(1 - r) \right]^2 a^2 / (1 + a^2)$ is deleted. It follows that

Corollary.

For arbitrary sample size and $|a| \leq 1/2$, an approximation to the structural distribution function of $R(\tau; \mu, \sigma)$ is

$$\tilde{F}_R \left(r | \tau, a(\underline{t}) \right) = 1 - \Phi(\eta) , \quad (5.11)$$

$$\text{where } \eta = \sqrt{\frac{n(2 + a^2)}{2(1 + a^2)}} \Phi^{-1}(1 - r) + \sqrt{\frac{2n - 3}{2 + a^2}} a .$$

Furthermore,

(i.) an approximate point estimate is

$$\hat{\tau}_{\hat{L}}[R(\tau; \mu, \sigma)] = \Phi \left\{ \sqrt{\frac{2(2n-3)(1+\alpha^2)}{(2+\alpha^2)[2(1+\alpha^2) + n(2+\alpha^2)]}} a \right\} \quad (5.12)$$

(ii.) and an approximate $(1-\alpha)$ -structural lower bound is

$$\hat{p}(1-\alpha) = 1 - \Phi \left\{ \sqrt{\frac{2(1+\alpha^2)}{n(2+\alpha^2)}} \left[\Phi^{-1}(1-\alpha) - \sqrt{\frac{2n-3}{2+\alpha^2}} a \right] \right\} \quad (5.13)$$

6. NUMERICAL EXAMPLE

To illustrate numerically some of the results of the previous sections, we consider the following data, given by Antle (1972), which have been generated from a lognormal distribution (with $\mu = 4.0$ and $\sigma = 0.5$):

26.96,	43.02,	57.99,	78.50,
28.06,	53.02,	60.39,	96.28,
39.52,	53.22,	65.92,	119.05,
41.01,	53.41,	66.99,	120.18,
41.41,	57.14,	69.73,	175.37.

Suppose the "mission time" is $\tau = 50.0$. Then based on the logarithmically transformed data,

$$\hat{\mu} = (1/n) \sum_{i=1}^n \ln t_i = 4.09688$$

$$\hat{\sigma} = \left[(1/n) \sum_{i=1}^n \left(\ln t_i - \hat{\mu} \right)^2 \right]^{1/2} = 0.46225$$

and

$$\alpha = 0.39991 \quad .$$

(i.) *Point estimates of $R(50; \mu, \sigma)$.*

Using Theorem 4.1, we find the mean, median, and mode of the structural density of $R(50; \mu, \sigma)$, which are the point estimates,

$$E[R(50; \mu, \sigma)] = 0.646 ,$$

$$r_{\text{median}} = 0.650 ,$$

$$r_{\text{mode}} = 0.657 .$$

An approximate point estimate based on equation (5.12) is

$$\approx E[R(50; \mu, \sigma)] = \Phi(.374) = 0.646 .$$

By way of comparison, the maximum likelihood estimate is

$$\hat{R}_{ML}(50; \mu, \sigma) = 1 - \Phi(-.400) = 0.655 ;$$

the minimum variance unbiased estimate is

$$\hat{R}_{UMVU}(50; \mu, \sigma) = 1 - I_{.454}(9, 9) = 0.650 .$$

(ii.) *Interval estimate of $R(50; \mu, \sigma)$.*

The 95% structural lower bound for $R(50; \mu, \sigma)$ is the iterative solution to the defining equation in Theorem 4.2. This solution,

$$\underline{r}(.95) = 0.501 ,$$

is also "numerically equivalent" to the 95% Bayes lower confidence bound under the improper prior $\pi(\mu, \sigma) \propto 1/\sigma$ and to the 95% classical lower confidence bound. An approximate 95% structural lower bound based on equation (5.13) is

$$\approx \underline{r}(.95) = 1 - \Phi(-.002) = 0.501 .$$

FIGURE 1. Structural Point Estimate for
the Reliability Function.

$$a = \frac{(n^{-1} \sum \ln t_i - \ln \tau)}{\sqrt{n^{-1} \sum (\ln t_i - n^{-1} \sum \ln t_i)^2}}$$

n is the sample size

FIGURE 2. 90% Structural Lower Bound for the Reliability Function.

$$\alpha = \frac{(n^{-1} \sum \ln t_i - \ln \tau)}{\sqrt{n^{-1} \sum (\ln t_i - n^{-1} \sum \ln t_i)^2}}$$

n is the sample size

FIGURE 3. 95% Structural Lower Bound for the Reliability Function.

$$a = \frac{(n^{-1} \sum \ln t_i - \ln \tau)}{\sqrt{n^{-1} \sum (\ln t_i - n^{-1} \sum \ln t_i)^2}}$$

n is the sample size

LIST OF REFERENCES

- Abramowitz, M. and Stegun, I. A. (1965). *Handbook of Mathematical Functions*, National Bureau of Standards, Applied Mathematics Series No. 55, U.S. Government Printing Office, Washington, D.C.
- Antle, C. E. (1972). Choice of Model for Reliability Studies and Related Topics, *Aerospace Research Laboratories TR 72-0108*, Wright-Patterson Air Force Base, Ohio.
- Bury, K. V. (1972). On the Reliability Analysis of a Two-Parameter Weibull Process, *INFOR*, Vol. 10, pp. 129-139.
- Bury, K. V. and Bernholtz, B. (1971). Life Testing: Structural Inference on the Exponential Model, *INFOR*, Vol. 9, pp. 148-160.
- Dumonceaux, R. and Antle, C. E. (1973). Discrimination Between the Log-Normal and the Weibull Distributions, *Technometrics*, Vol. 15, pp. 923-926.
- Epstein, B. (1947). The Mathematical Description of Certain Breakage Mechanisms Leading to the Logarithmico-Normal Distribution, *Journal of the Franklin Institute*, Vol. 244, pp. 471-477.
- Epstein, B. (1948). Statistical Aspects of Fracture Problems, *Journal of Applied Physics*, Vol. 19, pp. 140-147.
- Erdélyi, A., Magnus, W., Oberhettinger, F., and Tricomi, F. G. (1953). *Higher Transcendental Functions* (Bateman Manuscript Project), Vols. I and II, McGraw-Hill, New York.
- Feinlieb, M. (1960). A Method of Analyzing Log-Normally Distributed Survival Data With Incomplete Follow-Up, *Journal of the American Statistical Association*, Vol. 55, pp. 534-545.
- Fraser, D. A. S. (1968). *The Structure of Inference*, John Wiley and Sons, New York.
- Fraser, D. A. S. (1972). Bayes, Likelihood, or Structural, *Annals of Mathematical Statistics*, Vol. 43, pp. 777-790.
- Goldthwaite, L. R. (1961). Failure Rate Study for the Lognormal Lifetime Model, *Proceedings of the 7th National Symposium on Reliability and Quality Control*, pp. 208-213.
- Hora, R. B. and Buehler, R. J. (1966). Fiducial Theory and Invariant Estimation, *Annals of Mathematical Statistics*, Vol. 37, pp. 643-656.

- Johnson, N. L. and Kotz, S. (1970). *Continuous Univariate Distributions-1*, John Wiley and Sons, New York.
- Klimko, L. A., Antle, C. E., and Rademaker, A. (1975). Tests for Normality Versus Lognormality, *Communications in Statistics*, Vol. 4, pp. 1009-1019.
- Kotz, S. (1973). Normality vs. Lognormality with Applications, *Communications in Statistics*, Vol. 1, pp. 113-132.
- Lebedev, N. N. (1965). *Special Functions and Their Applications*, Prentice-Hall, Englewood Cliffs, New Jersey.
- Lieberman, G. J. and Resnikoff, G. J. (1955). Sampling Plans for Inspection by Variables, *Journal of the American Statistical Association*, Vol. 50, pp. 457-516.
- Nowick, A. S. and Berry, B. S. (1961). Lognormal Distribution Function for Describing Anelastic and Other Relaxation Processes, *IBM Journal of Research and Development*, Vol. 5, pp. 297-311, *Ibid.*, pp. 312-320.
- Olver, F. W. J. (1959). Uniform Asymptotic Expansions for Weber Parabolic Cylinder Functions of Large Orders, *Journal of Research of the National Bureau of Standards-B*, Vol. 63, pp. 131-169.
- Owen, D. B. (1962). *Handbook of Statistical Tables*, Addison-Wesley, Reading, Massachusetts.
- Zacks, S. (1971). *The Theory of Statistical Inference*, John Wiley and Sons, New York.
- Zacks, S. and Milton, R. C. (1971). Mean Square Errors of the Best Unbiased and Maximum Likelihood Estimators of Tail Probabilities in Normal Distributions, *Journal of the American Statistical Association*, Vol. 66, pp. 590-593.