ON THE EXISTENCE OF ZEROS
OF LYAPUNOV-MONOTONE OPERATORS

by

V. Lakshmikantham and S. Leela[†]

Technical Report No. 19

February 1975

[†] State University College, Geneseo, New York 14454.

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1. Introduction.

Consider a nonlinear operator T from a Banach space into itself. The study of the existence of zeros of T plays an important role in yielding fixed points of nonlinear operators. The operator T has a zero if and only if the initial value problem

$$(1.1) \quad x' + Tx = 0, \quad x(0) = x_0,$$

has a constant solution. If T is a monotone operator then (1.1) has a unique solution $x(t, x_0)$ defined on $[0, \infty)$ and the solution operator $U(t)x = x(t, x_0)$ is nonexpansive for all $t \geq 0$. Imposing further assumptions one can show that $U(t)$ must have a common fixed point and that fixed point is the desired zero of T . Thus one can use the theory of differential equations and some known fixed point theorems on the solution operator to obtain the existence of zeros of T . See for example [1,3,4, 5,9,11].

In this paper, we introduce the notion of Lyapunov-monotone operators in terms of several Lyapunov-like functions and utilizing certain results in abstract cones that are recently proved [6], we establish the existence of zeros of such operators. This leads us to work with generalized Banach spaces which offer a flexible technic. The results obtained are so general that when we employ as a candidate a generalized norm, we still get results more general than the known ones [8,9].

2. Preliminary results.

We shall use vectorial inequalities freely with the understanding that the same inequalities hold between their corresponding components. Let E be a real vector space. A generalized norm for E is a mapping $||\cdot||_G: E \rightarrow \mathbb{R}_+^p$ denoted by

$$||x||_G = (\alpha_1(x), \alpha_2(x), \dots, \alpha_p(x)), \quad x \in E,$$

such that

- (i) $||x||_G \geq 0$,
- (ii) $||x||_G = 0$ if and only if $x = 0$,
- (iii) $||\lambda x||_G = |\lambda| ||x||_G, \quad \lambda \in \mathbb{R}$,
- (iv) $||x+y||_G \leq ||x||_G + ||y||_G$.

For $x \in E$ and $\epsilon \in \mathbb{R}_+^p, \epsilon > 0$, we let $B(x, \epsilon) = [y \in E: ||y-x||_G < \epsilon]$. Then $[B(x, \epsilon): x \in E, \epsilon \in \mathbb{R}_+^p, \epsilon > 0]$ is a basis for a topology on E .

It is not difficult to see that every generalized normed space $(E, ||\cdot||_G)$ has an equivalent (ordinary) norm. For purely algebraic and topological considerations, it is immaterial whether we view E as a generalized normed space or an ordinary normed space. Concepts such as convexity, closure, completeness and compactness remain the same. We do, however, have more flexibility in working with generalized normed spaces. See for example [2].

Let E be a real Banach space. A set $K \subseteq E$ is called a cone if

- (i) K is closed,
- (ii) if $u, v \in K$ then $\alpha u + \beta v \in K$ for all $\alpha, \beta \geq 0$,

(iii) of each pair of vectors $u, -u$ at least one does not belong to K , provided $u \neq 0$ where 0 is the zero element of E . We say that $u \geq v$ if and only if $u - v \in K$. A cone K is said to be normal if there exists a $\delta > 0$ such that $\|e_1 + e_2\| \geq \delta$ for $e_1, e_2 \in K$ and $\|e_1\| = \|e_2\| = 1$. A cone is called regular if each bounded monotonic sequence has a limit.

The following results will be useful in our subsequent discussion.

Theorem 2.1. Let ψ be a monotone mapping of a cone segment $0 \leq u \leq u_0$ into itself and let either (i) ψ be upper semi-continuous from the right, K is regular or (ii) ψ be completely continuous, K is normal (or both). Then the sequence of iterates $\{\psi^n u_0\}$ is decreasing and converges to a fixed point w of ψ , that is $\psi w = w$. Moreover, $v \leq \psi v$ implies that $v \leq w$ where w is the maximal fixed point of ψ in the cone segment.

Let X be a set and K be a cone in E . A function $\rho: X \times X \rightarrow K$ is said to be a k -metric on X if

- (i) $\rho(x, y) = \rho(y, x)$,
- (ii) $\rho(x, y) = 0$ if and only if $x = y$,
- (iii) $\rho(x, z) \leq \rho(x, y) + \rho(y, z)$, $x, y, z \in X$.

A sequence $\{x_n\}$ in a k -metric space is said to be Cauchy sequence if $\lim_{m, n \rightarrow \infty} \|\rho(x_n, x_m)\| = 0$. A k -metric space is complete if every Cauchy's sequence is convergent.

Theorem 2.2. Let ψ satisfy the assumptions of Theorem 2.2. Suppose further that $\psi u_0 < u_0$ and $\psi u = u$ if and only if $u = 0$. Let X be a complete k -metric space and let T be a mapping of X into itself such that

$$\rho(Tx, Ty) \leq \psi \rho(x, y), \text{ for } \rho(x, y) \leq u_0$$

Then if $x_0 \in X$ with $\rho(Tx_0, x_0) \leq u_0$, the sequence of iterates $\{T^n x_0\}$ converges to a fixed point w of T and that w is unique in $B = [y \in X : \rho(y, x_0) \leq u_0]$.

For more details concerning the k -metric and for the proofs of Theorems 2.1 and 2.2 see [6].

We shall be employing Theorems 2.1 and 2.2 subsequently in the special case where $K = \mathbb{R}_+^p$ and the metric $\rho(x, y) = \|x - y\|_G$.

3. Statement of main results.

Let E be a generalized Banach space with the norm $\|\cdot\|_G$ and let T be a continuous map from E into E . For any $V \in C[E \times E, \mathbb{R}_+^p]$, we define

$$(3.1) \quad D^+ V(x, y) = \limsup_{h \rightarrow 0^+} \frac{1}{h} \left[V(x + hTx, y + hTy) - V(x, y) \right].$$

We need the following assumptions to proceed.

(H₀) For each $q \in \mathbb{R}_+^p$, $q > 0$, $g_q \in C[\mathbb{R}_+^p, \mathbb{R}_+^p]$, $g_q(u)$ is quasimonotone nondecreasing in u , $g_q(0) = 0$ and $u \equiv 0$ is the only solution of the

differential system

$$(3.2) \quad u' = g_q(u), \quad u(0) = 0.$$

(H₁) $g \in C[R_+^p, R^p]$, $g(u)$ is quasimonotone nondecreasing in u , $g(0) = 0$ and $u \equiv 0$ is the only solution of the differential system

$$(3.3) \quad u' = g(u), \quad u(0) = 0.$$

Definition. Let $V \in C[E \times E, R_+^p]$ and the assumption (H₁) hold. If for $x, y \in E$,

$$(3.4) \quad D^+V(x, y) + g(V(x, y)) \geq 0,$$

then we shall say that T is a strongly Lyapunov-monotone operator on E .

If, in (3.4) either $g \in C[R_+^p, R_+^p]$ or $g(u) \equiv 0$, then T is said to be Lyapunov-monotone operator on E . If (3.4) is weakened to

$$(3.5) \quad D^+V(x, y) + g_q(V(x, y)) \geq 0, \quad x, y \in B[q],$$

where g_q satisfies the hypothesis (H) for every $q \in R_+^p$, $q > 0$ and $B[q] = \{x \in E : \|x\|_G \leq q\}$, then T is said to be strongly Lyapunov-monotone operator on $B[q]$. In the special case when $V(x, y) = \|x - y\|_G$ or $V(x, y)$ is a metric, then the operator T is said to be strongly generalized-monotone on E , generalized-monotone on E and strongly generalized-monotone on $B[q]$ respectively.

For convenience, we shall list a few assumptions below.

(H₂) $V \in C[E \times E, R_+^p]$, $V(x, y) = 0$, $V(x, y) > 0$ if $x \neq y$, $x, y \in E$ and if $\{x_n\}$, $\{y_n\}$ are sequences in E such that $\lim_{n \rightarrow \infty} V(x_n, y_n) = 0$, then $\lim_{n \rightarrow \infty} \|x_n - y_n\|_G = 0$.

(H₃) for each $(x, y) \in E \times E$, there is a neighbourhood $N(x, y)$ and a positive $n \times n$ matrix $L = L(x, y)$ such that

$$|V(x_1, y_1) - V(x_2, y_2)| \leq L[||x_1 - x_2||_G + ||y_1 - y_2||_G],$$

for all $(x_1, y_1), (x_2, y_2) \in N(x, y)$.

(H₄) for every $q \in \mathbb{R}_+^p$, $q > 0$, there is a nonnegative $n \times n$ matrix $A(q)$ such that for $x, y \in B[q]$

$$||x - y||_G \leq V(x, y) \leq A(q) ||x - y||_G.$$

(H₅) $||Tx||_G \rightarrow \infty$ as $||x||_G \rightarrow \infty$.

(H₆) The maximal solution $r(t, u_0)$ of the differential system

$$(3.6) \quad u' = g(u) + ||T0||_G, \quad u(0) = u_0 \geq 0,$$

exists on $[0, \infty)$ and there is a $q_0 \in \mathbb{R}_+^p$, $q_0 > 0$ such that

$$(3.7) \quad r(t, u_0) \leq q_0, \quad t \geq 0 \quad \text{if} \quad 0 \leq u_0 \leq q_0.$$

Furthermore, there exists a positive number τ such that

$$(3.8) \quad r(\tau, u_0) \leq u_0, \quad \text{for every} \quad 0 \leq u_0 \leq 2q_0,$$

with the strict inequality holding at $u_0 = 2q_0$, where $r(t, u_0)$ is the maximal solution of

$$(3.9) \quad u' = g(u), \quad u(0) = u_0 \geq 0.$$

(H₇) for every $q \in \mathbb{R}_+^p$, $q > 0$, the maximal solution $r_q(t, u_0)$ of

$$(3.10) \quad u' = g_q(u), \quad u(0) = u_0 \geq 0,$$

exists on $[0, \infty)$ and there is a positive number $\tau > 0$ such that

$$(3.11) \quad r_q(\tau, A(q)u_0) \leq u_0 \quad \text{for every } 0 \leq u_0 \leq 2q,$$

with the strict inequality holding at $u_0 = 2q$.

We are now in a position to state our main results.

Theorem 3.1. Assume that

- (a) T is a Lyapunov-monotone operator on E with $g(u) \equiv 0$ in (3.4) and (H_2) , (H_3) hold;
- (b) T is a strongly Lyapunov-monotone operator on $B[q]$ for every $q \in \mathbb{R}_+^p$, $q > 0$ and (H_4) , (H_5) and (H_7) are satisfied.

Then the operator T has a unique zero in E .

If $V(x, y) = \|x - y\|_G$ or $V(x, y)$ is a metric, Theorem 3.1 can be considerably simplified. This we state as a next result.

Theorem 3.2. Assume that T is a strongly generalized-monotone operator on E and that the hypothesis (H_6) holds. Then there is a $q_0 \in \mathbb{R}_+^p$, $q_0 > 0$ such that T has a unique zero in $B[q_0]$.

Remark. One can use a single Lyapunov-like function $V(x, y)$ to define Lyapunov-monotone operators in which case g must be a scalar function. In

addition, if we choose $V(x,y) = ||x-y||$, where $||\cdot||$ is the ordinary norm, then our definitions reduce to the known definitions of monotone operators. The special cases of Theorems 3.1 and 3.2 that result with the foregoing observation are interesting in themselves. See [8,9] for results parallel to these special cases.

4. Auxiliary results.

We shall first prove an existence result for the initial value problem

$$(4.1) \quad x' + Tx = 0, \quad x(0) = x_0 \in E.$$

Theorem 4.1. Let T be a Lyapunov-monotone operator on E and (H_2) , (H_3) hold. Then for each $x_0 \in E$, there is a unique solution $x(t, x_0)$ of (4.1) defined on $[0, \infty)$.

To prove Theorem 4.1 we require the following results.

Theorem 4.2. Let (a) $g \in C[R^+ \times R_+^p, R^p]$ and $g(t, u)$ be quasi-monotone nondecreasing in u for each t ; (b) $[t_0, t_0 + \alpha)$ be the largest interval of existence of the maximal solution $x(t, t_0, u_0)$ of

$$u' = g(t, u), \quad u(t_0) = u_0 \geq 0;$$

(c) $m \in C[R^+, R_+^p]$ and for any fixed Dini derivative, the inequality

$$Dm(t) \leq g(t, m(t)), \quad t \geq t_0,$$

holds. Then $m(t_0) \leq u_0$ implies

$$m(t) \leq r(t, t_0, u_0), \quad t \in [t_0, t_0 + \alpha).$$

Theorem 4.3. Let the assumptions (a) and (b) of Theorem 4.2 hold. Suppose that $[t_0, t_1] \subset [t_0, t_0 + \alpha)$. Then there exists a $\varepsilon_0 \in R_+^p$, $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, the maximal solution $r(t, t_0, u_0, \varepsilon)$ of

$$u' = g(t, u) + \varepsilon, \quad u(t_0) = u_0 + \varepsilon,$$

exists on $[t_0, t_1]$ and $\lim_{\varepsilon \rightarrow 0} r(t, t_0, u_0, \varepsilon) = r(t, t_0, u_0)$ uniformly on $[t_0, t_1]$.

For a proof of Theorems 4.2 and 4.3 see [7].

Lemma 4.1. Let $\alpha > 0$, $b \in R_+^p$, $b > 0$ be chosen such that $\|Tx\|_G \leq \frac{b}{\alpha} \equiv M$ on $B[x_0, b]$. Then for each positive integer n , there is a positive integer $N = N(n)$, a partition $\{t_i^n\}_{i=0}^N$ of $[0, \alpha]$, and a function x_n from $[0, \alpha]$ into $B[x_0, b]$ such that

$$(i) \quad |t_{i+1}^n - t_i^n| \leq \frac{1}{n} \quad \text{for each } 1 \leq i \leq N;$$

$$(ii) \quad \|x_n(t) - x_n(s)\|_G \leq M(t-s), \quad t, s \in [0, \alpha];$$

$$(iii) \quad \text{if } t \in (t_{i-1}^n, t_i^n), \text{ then the left derivative } (x_n)'_-(t) \text{ exists and } (x_n)'_-(t) + Tx_n(t_{i-1}^n) = 0 \text{ for each } 1 \leq i \leq N;$$

$$(iv) \quad \text{if } \|x - x_n(t_{i-1}^n)\|_G \leq M(t_i^n - t_{i-1}^n), \text{ then } \|Tx - Tx_n(t_{i-1}^n)\|_G \leq \varepsilon_n$$

where $\{\varepsilon_n\}_{n=1}^\infty$ is a sequence such that $\varepsilon_n \in R_+^p$, $\varepsilon_n > 0$

and $\lim_{n \rightarrow \infty} \varepsilon_n = 0$.

Lemma 4.2. Suppose that $\lim_{n \rightarrow \infty} x_n(t) = x(t)$ for each $t \in [0, a]$. Then $x(t)$ is a solution of the problem (4.1).

The proofs of Lemmas 4.1 and 4.2 are similar to the proofs of Lemmas 4 and 6 in [8] with appropriate modifications and hence omitted.

Proof of Theorem 4.1. Let $x_0 \in E$ and let $\{x_n(t)\}$ be the sequence of approximate solutions constructed in Lemma 4.1 such that $x_n(0) = x_0$. Let m, n be positive integers and let $m(t) = V(x_n(t), x_m(t))$, $t \in [0, a]$. If $t \in (t_{i-1}^n, t_i^n) \cap (t_{j-1}^m, t_j^m)$, then using (H_3) , where $L = L(x_0, x_0)$, we obtain

$$\begin{aligned}
 (4.2) \quad D_m(t) &= \liminf_{h \rightarrow 0^+} \frac{1}{h} \left[V(x_n(t), x_m(t)) - V(x_n(t-h), x_m(t-h)) \right] \\
 &\leq \liminf_{h \rightarrow 0^+} \frac{1}{h} \left[V(x_n(t), x_m(t)) - V(x_n(t) - hTx_n(t), x_m(t) - hTx_m(t)) \right] \\
 &\quad + \liminf_{h \rightarrow 0^+} L \left[\frac{\|x_n(t) - x_n(t-h) - Tx_n(t)\|_G}{h} + \frac{\|x_m(t) - x_m(t-h) - Tx_m(t)\|_G}{h} \right] \\
 &\leq g(V(x_n(t), x_m(t))) + L \left[\|Tx_n(t_{i-1}^n) - Tx_n(t)\|_G + \|Tx_m(t_{j-1}^m) - Tx_m(t)\|_G \right] \\
 &\leq g(m(t)) + L(\epsilon_n + \epsilon_m).
 \end{aligned}$$

Here we have used the relation (iv) of Lemma 4.1 and the fact that (3.4) implies that $\liminf_{h \rightarrow 0^+} \frac{1}{h} [V(x, y) - V(x - hTx, y - hTy)] \leq g(V(x, y))$. Since

$m(0) = 0$, the vector differential inequality, gives by Theorem 4.2 the estimate

$$m(t) \leq r_{n,m}(t, 0), \quad t \in [0, a]$$

where $r_{n,m}(t, 0)$ is the maximal solution of

$$u' = g(u) + L(\epsilon_n + \epsilon_m), \quad u(0) = 0.$$

By Theorem 4.3, $\lim_{n,m \rightarrow \infty} r_{n,m}(t, 0) = r(t, 0)$ uniformly on $[0, a]$ where $r(t, 0)$ is the maximal solution of (3.3). But, by (H_1) , $r(t, 0) \equiv 0$ on $[0, a]$ and hence $\lim_{n,m \rightarrow \infty} V(x_n(t), x_m(t)) = 0$. This implies by (H_2) , the sequence $\{x_n(t)\}_{n=1}^{\infty}$ is Cauchy and since this sequence is equicontinuous, $\lim_{n \rightarrow \infty} x_n(t) = x(t)$ uniformly on $[0, a]$. It now follows by Lemma 4.2 that $x(t)$ is a solution of the initial value problem (4.1) on $[0, a)$.

If $y(t)$ is another solution of (4.1) on $[0, a)$, then setting $m(t) = V(x(t), y(t))$ and proceeding as before, we get

$$D_- m(t) \leq g(m(t)), \quad t \in [0, a),$$

and $m(t) \leq r(t, 0)$, $t \in [0, a)$, where $r(t, 0)$ is the maximal solution of (3.3). Consequently, we conclude $x(t) \equiv y(t)$ on $[0, a)$, by (H_1) and (H_2) . This establishes local existence and uniqueness of solutions of (4.1).

To show that solutions exist on $[0, \infty)$, suppose that $x(t)$ is a solution of (4.1) which is defined only in $[0, a)$ with $a < \infty$. If $0 < h < a$, set $m(t) = V(x(t+h), x(t))$ for $t \in [0, a-h)$. Then it is easy to obtain the estimate

$$(4.3) \quad V(x(t+h), x(t)) \leq r(t, V(x(h), x(0))), \quad t \in [0, a-h],$$

where $r(t, u_0)$ is the maximal solution of (3.9). Since $\lim_{h \rightarrow 0} V(x(h), x(0)) = 0$, it follows by $(H_1), (H_2), (4.3)$ and Theorem 4.3 that $\lim_{t \rightarrow a^-} x(t)$ exists and this implies that $x(t)$ can be defined on $[0, \infty)$. The proof is complete.

Corollary 4.1. Let T be a generalized monotone operator on E . Then the conclusion of Theorem 4.1 is valid.

Theorem 4.4. Let T be a Lyapunov-monotone operator on E with $g(u) \equiv 0$ in (3.4) and $(H_2), (H_3), (H_4)$ and (H_5) hold. Then all solutions $x(t, x_0)$ of (4.1) are bounded on $[0, \infty)$.

Proof. Since all solutions $x(t, x_0)$ of (4.1) exist on $[0, \infty)$ by Theorem 4.1, setting $m(t) = V(x(t, x_0), x(t, y_0))$, we obtain the inequality

$$m(t) \leq m(0), \quad t \in [0, \infty).$$

If $x_0, y_0 \in B[q]$, this gives by (H_4) , the estimate

$$(4.4) \quad \|x(t, x_0) - x(t, y_0)\|_G \leq A(q) \|x_0 - y_0\|_G, \quad t \in [0, \infty).$$

Since by uniqueness, $x(t+h, x_0) = x(t, y_0)$, where $y_0 = x(h, x_0)$ for $h > 0$ sufficiently small, it follows from (4.4)

$$\|x(t, x_0) - x(t+h, x_0)\|_G \leq A(q) \|x_0 - x(h, x_0)\|_G.$$

Dividing by h and taking limit as $h \rightarrow 0^+$ yields the estimate

$$||Tx(t, x_0)||_G \leq A(q) ||Tx_0||_G,$$

which by (H_5) shows that $x(t, x_0)$ is bounded on $[0, \infty)$. Since the arguments hold for every $q \in R_+^p$, $q > 0$, the proof is complete.

Remark. Theorem 4.1 includes the existence and uniqueness results in [7,9,10], while Corollary 4.1 is a generalization of the results in [8]. Theorem 4.4 isolates and generalizes the ideas in [9].

5. Proofs of main results.

We are now in a position to prove our main results.

Proof of Theorem 3.1. Let $x_0 \in E$ and let $q \in R_+^p$, $q > 0$ be such that $x(t, x_0), x(t, y_0) \in B[q]$, $t \geq 0$. This is possible by Theorems 4.1 and 4.4. Setting $m(t) = V(x(t, x_0), x(t, y_0))$, we obtain, because of the assumption (b), the inequality

$$D_- m(t) \leq g_q(m(t)), \quad t \geq 0,$$

which implies

$$m(t) \leq r_q(t, m(0)), \quad t \geq 0,$$

where $r_q(t, u_0)$ is the maximal solution of (3.10). By (H_4) , we then have

$$(5.1) \quad ||x(t, x_0) - x(t, y_0)||_G \leq r_q[t, A(q) ||x_0 - y_0||_G], \quad t \geq 0.$$

For each $(t, x_0) \in R^+ \times E$, let us define the solution operator U by

$U(t)x_0 = x(t, x_0)$. Then U is a semigroup of nonlinear operators on E . Let us also define $\psi_\tau u = r_q[\tau, A(q)u]$ where $\tau > 0$ is the number given by (H_7) . We then obtain from (5.1), the relation

$$(5.2) \quad \|U(\tau)x_0 - U(\tau)y_0\|_G \leq \psi_\tau \|x_0 - y_0\|_G.$$

Consider the cone $K = R_+^p$. By (H_7) it follows that ψ_τ maps the cone segment $0 \leq u \leq 2q$ into itself and $\psi_\tau 2q < 2q$. Also by (H_0) , $u = \psi_\tau u$ has the only fixed point $u = 0$. Furthermore, if $v \leq w$, we know that $r_q(t, v) \leq r_q(t, w)$, where $r_q(t, u_0)$ is the maximal solution of (3.10). This implies that ψ_τ is monotone. Clearly ψ_τ is completely continuous. Considering E as a generalized metric space with the metric $\|x - y\|_G$ we can now conclude from (5.2) and Theorem 2.2 that there exists a unique $x_\tau \in B[2q]$ such that $U(\tau)x_\tau = x_\tau$ for every $q > 0$. Since $x_\tau = x(0, x_\tau) = x(\tau, x_\tau)$ and (4.1) is uniquely solvable, it follows that $x(t, x_\tau)$ is a τ periodic solution. Now using the semigroup property of U and (5.2) we have

$$(5.3) \quad \|x(t, x_\tau) - x_\tau\|_G = \|x(t+\tau, x_\tau) - x_\tau\|_G = \|U(\tau)x(t, x_\tau) - U(\tau)x_\tau\|_G \\ \leq \psi_\tau \|x(t, x_\tau) - x_\tau\|_G.$$

Letting $v(t) = \|x(t, x_\tau) - x_\tau\|_G$ and noting that $v \leq \|x(t, x_\tau)\|_G + \|x_\tau\|_G \leq 2q$, the relation (5.3) implies that $v(t) \leq \psi_\tau v(t)$, for each $t \geq 0$. Hence by comparison Theorem 2.2 we get $v(t) \leq r$ where r is the maximal fixed point of $u = \psi_\tau u$ in the cone segment $0 \leq u \leq 2q$. But $u = 0$ is the only fixed point of $u = \psi_\tau u$. Thus $v(t) = 0$ for all $t \geq 0$ and this implies that $U(t)x_\tau = x(t, x_\tau) = x_\tau$ for all $t \geq 0$. It therefore

follows from (4.1) that $Tx_\tau = 0$ and this completes the proof of Theorem 3.1.

We shall next give the proof of Theorem 3.2.

Proof of Theorem 3.2. By Corollary 4.1, for every $x_0 \in E$, there exists a unique solution $x(t, x_0)$ of (4.1) defined on $[0, \infty)$. Let $m(t) = \|x(t, x_0)\|_G$. Then, using the fact that T is strongly generalized monotone operator, we get

$$\begin{aligned} D_- m(t) &\leq \liminf_{h \rightarrow 0^+} \frac{1}{h} \left[\|x(t, x_0)\|_G - \|x(t, x_0) - hTx(t, x_0) + hT0\|_G \right] \\ &\quad + \liminf_{h \rightarrow 0^+} \frac{1}{h} \left[\|x(t, x_0) - x(t-h, x_0) - hTx(t, x_0)\|_G \right] + \|T0\|_G \\ &\leq g(m(t)) + \|T0\|_G, \quad t \geq 0. \end{aligned}$$

Consequently, Theorem 4.2 yields $m(t) \leq r(t, \|x_0\|_G)$, $t \geq 0$, where $r(t, u_0)$ is the maximal solution of (3.6). Thus by (3.7) of (H_6) ,

$$(5.4) \quad \|x(t, x_0)\|_G \leq q_0 \quad \text{for all } t \geq 0, \text{ if } \|x_0\| \leq q_0.$$

Defining, as before, $U(t)x_0 = x(t, x_0)$, it is clear from (5.4) that $U(t): B[q_0] \rightarrow B[q_0]$ for each $t \geq 0$.

Corresponding to (5.1), in the present case, we arrive at

$$\|x(t, x_0) - x(t, y_0)\| \leq r(t, \|x_0 - y_0\|), \quad t \geq 0,$$

where $r(t, u_0)$ is the maximal solution of (3.9). We now set $\psi_\tau u = r(\tau, u)$ and observe that all the hypotheses of Theorem 2.3 are satisfied with $K = R_+^p$

in the cone segment $0 \leq u \leq 2q_0$ in view of (H_6) and using similar arguments as before. We can conclude by Theorem 2.2 that there exists a unique $x_\tau \in B[q_0]$ such that $U(\tau)x_\tau = x_\tau$. The rest of the proof of showing that $x(t, x_\tau)$ is τ periodic, $x(t, x_\tau) = x_\tau$ for all $t \geq 0$ and $Tx_\tau = 0$ is similar to the last part of the proof of Theorem 3.1. The proof is therefore complete.

Corollary 5.1. Suppose that for $x, y \in E$,

$$\limsup_{h \rightarrow 0^+} \frac{1}{h} \left[\|x - y + h(Tx - Ty)\| - \|x - y\| \right] \geq g(\|x - y\|)$$

where $\|\cdot\|$ is the ordinary norm, $g \in C[R^+, R^+]$, $g(0) = 0$, $g(u) > 0$, $g(u) \rightarrow \infty$ as $u \rightarrow \infty$ and $\liminf_{u \rightarrow 0} \frac{g(u)}{u} > 0$. Then there exists a unique $x^* \in E$ such that $Tx^* = 0$.

Proof. It is enough to show, by Theorem 3.2, that (H_6) is satisfied. It can be easily checked that with $q_0 = \inf [u > 0 : g(u) > \|T(0)\|]$, in (u, ∞) , (3.7) holds. By assumption, there exists a $c > 0$ and $r_0 > 0$ such that $g(u) \geq cu$ in $[0, r_0]$. If $\gamma = \inf [g(u) : u \geq r_0] > 0$, then, it can be shown that $r(\tau, u_0) \leq \frac{u_0}{2}$ for $0 \leq u_0 \leq 2q$ where $\tau = \gamma^{-1}(2q_0 - r_0) + c^{-1} \log 2$.

Remark. The details omitted in the proof of Corollary 5.1 may be seen in [5] where an analogous result is proved. However, the operator T is defined over a closed subset of E and the assumptions are imposed in terms of a generalized pairing rather than norm.

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