A NEW APPROACH TO THE METHOD
OF NONLINEAR VARIATION OF PARAMETERS

by

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1. Introduction.

As is well-known [3] the method of variation of parameters is a very useful tool in studying the properties of solutions of perturbed differential equations. Extending the classical variation of constants formula for linear systems, Alekseev [1] obtained a variation of constants formula for nonlinear systems. In [2], Lakshmikantham developed a nonlinear variation of constants formula for a scalar differential equation whose unperturbed term is of variable separable type, under rather mild assumptions.

In this paper, we wish to study the general problem under weaker assumptions. Our approach is parallel to the classical one and yields two different formulas, one of which is shown to be equivalent to the Alekseev's formula under additional restrictions. Also, the formula given in [2] is shown to be a consequence of our results. To illustrate the flexibility of the results derived, we give as applications results on stability and asymptotic equilibrium.

2. Consider the differential system

$$(2.1) \quad x' = f(t, x), \quad x(t_0) = x_0, \quad t_0 \in R^+, \quad x_0 \in R^n$$

and its perturbed system

$$(2.2) \quad y' = f(t, y) + F(t, y), \quad y(t_0) = x_0,$$

when $f, F \in C[R^+ \times R^n, R^n]$.

Theorem 2.1. Suppose that the system (2.1) admits unique solutions $x(t, t_0, x_0)$.

Suppose, also, that $\phi(t, t_0, x_0) \equiv \frac{\partial x}{\partial x_0}(t, t_0, x_0)$ exists and is continuous for all $t \geq t_0$ and that $\phi^{-1}(t, t_0, x_0)$ exists for all $t \geq t_0$. If $\phi(t)$ is a solution of (2.5) then any solution $y(t, t_0, x_0)$ of (2.2) satisfies the relation

$$(2.3) \quad y(t, t_0, x_0) = x\left(t, t_0, x_0 + \int_{t_0}^t \phi^{-1}(s, t_0, \phi(s)) F(s, y(s, t_0, x_0)) ds\right),$$

as far as $\phi(t)$ exists to the right of t_0 .

Proof. Let $x(t, t_0, x_0)$ be any solution of (2.1) existing for $t \geq t_0$.

The method of variation of parameters requires determining a function $\phi(t)$ so that

$$(2.4) \quad y(t, t_0, x_0) = x(t, t_0, \phi(t)), \quad \phi(t_0) = x_0,$$

is a solution of (2.2). Differentiating with respect to t yields

$$y'(t, t_0, x_0) = x'(t, t_0, \phi(t)) + \frac{\partial x}{\partial x_0}(t, t_0, \phi(t)) \phi'(t).$$

Thus

$$\begin{aligned} f(t, y(t, t_0, x_0)) + F(t, y(t, t_0, x_0)) &= f(t, x(t, t_0, \phi(t))) \\ &+ \phi(t, t_0, \phi(t))\phi'(t) \end{aligned}$$

which because of (2.4) and the fact that $\bar{\Phi}^{-1}(t, t_0, x_0)$ exists, reduces to

$$(2.5) \quad \phi'(t) = \bar{\Phi}^{-1}(t, t_0, \phi(t))F(t, x(t, t_0, \phi(t))), \quad \phi(t_0) = x_0.$$

Solutions of (2.5) then determine $\phi(t)$.

Consequently, if $\phi(t)$ is a solution of (2.5), then $y(t, t_0, x_0)$ given by (2.4) is a solution of (2.2). From (2.5) $\phi(t)$ must satisfy the integral equation

$$\phi(t) = x_0 + \int_{t_0}^t \bar{\Phi}^{-1}(s, t_0, \phi(s))F(s, x(s, t_0, \phi(s)))ds,$$

which establishes the relation (2.3). Notice that $y(t, t_0, x_0)$ exists for those values of $t \geq t_0$ for which $\phi(t)$ exists. ||

Corollary 2.1. If $f(t, x) = A(t)x$, then $x(t, t_0, x_0) = \Phi(t, t_0)x_0$, where $\Phi(t, t_0)$ is the fundamental matrix solution of $x' = A(t)x$ with $\Phi(t_0, t_0) = I$ (identity matrix). Then the relation (2.3) yields a well-known form

$$(2.6) \quad y(t, t_0, x_0) = x(t, t_0, x_0) + \int_{t_0}^t \Phi(t, t_0) \bar{\Phi}^{-1}(s, t_0)F(s, y(s, t_0, x_0))ds.$$

The question whether the relation (2.3) can be obtained in the form (2.6) is answered by the next result.

Theorem 2.2. Under the assumptions of Theorem 2.1, the following relation is also valid

$$(2.7) \quad y(t, t_0, x_0) = x(t, t_0, x_0) + \int_{t_0}^t \Phi(t, t_0, \phi(s)) \Phi^{-1}(s, t_0, \phi(s)) \cdot F(s, y(s, t_0, x_0)) ds,$$

where $\phi(t)$ is any solution of (2.5).

Proof. For $t_0 \leq s \leq t$ we have

$$\frac{d}{ds} x(t, t_0, \phi(s)) = \frac{\partial x}{\partial x_0}(t, t_0, \phi(s)) \phi'(s) = \Phi(t, t_0, \phi(s)) \phi'(s).$$

Integrating from t_0 to t yields

$$x(t, t_0, \phi(t)) = x(t, t_0, x_0) + \int_{t_0}^t \Phi(t, t_0, \phi(s)) \phi'(s) ds.$$

If $\phi(t)$ is any solution of (2.5) then the result (2.7) follows immediately, because of (2.4) and (2.5). $\square$

Remark. The relations (2.3) and (2.7) offer two different forms of the variation of parameters formula for solutions $y(t, t_0, x_0)$ of (2.2). This offers flexibility in applications as we shall see later.

Corollary 2.2. Consider the scalar differential equations

$$(2.8) \quad x' = \lambda(t)g(x), \quad x(t_0) = x_0, \quad \text{and}$$

$$(2.9) \quad y' = \lambda(t)g(y) + R(t,y), \quad y(t_0) = x_0,$$

where $\lambda \in C[R^+, R]$, $g \in C[R^+, R^+]$, $g(0) = 0$, $g(u) > 0$, $u > 0$ and $R \in C[R^+ \times R^+, R]$. Suppose that $\int_{t_0}^t \frac{ds}{g(s)} = \infty$. Then any solution $y(t, t_0, x_0)$ of (2.9) may be expressed in the two forms

$$(2.10) \quad y(t, t_0, x_0) = G^{-1} \left[G(\phi(t)) + \int_{t_0}^t \lambda(s) ds \right]$$

and

$$(2.11) \quad y(t, t_0, x_0) = G^{-1} \left[\int_{t_0}^t \lambda(s) ds + G \left\{ x_0 + \int_{t_0}^t \frac{g(G^{-1}[G(y(s)) - \int_{t_0}^s \lambda(\xi) d\xi])}{g(y(s))} R(s, y(s)) ds \right\} \right]$$

where $\frac{dG(x)}{dx} = \frac{1}{g(x)}$ and $\phi(t)$ is any solution of

$$(2.12) \quad \phi'(t) = \frac{g(\phi(t))R\left(t, G^{-1}\left(G(\phi(t)) + \int_{t_0}^t \lambda(s) ds\right)\right)}{g\left[G^{-1}\left\{G(\phi(t)) + \int_{t_0}^t \lambda(s) ds\right\}\right]}, \quad \phi(t_0) = x_0.$$

Proof. We know that solutions of (2.8) are unique by the assumed Osgood's conditions and are given by

$$G(x(t, t_0, x_0)) = G(x_0) + \int_{t_0}^t \lambda(s) ds.$$

Hence $\frac{\partial G}{\partial x} \left(x(t, t_0, x_0) \right) \frac{\partial x}{\partial x_0} (t, t_0, x_0) = \frac{\partial G}{\partial x_0} (x_0)$, and this implies by

the definition of G , that

$$\Phi(t, t_0, x_0) \equiv \frac{\partial x}{\partial x_0} (t, t_0, x_0) = \frac{g(x(t, t_0, x_0))}{g(x_0)},$$

proving the existence of $\Phi(t, t_0, x_0)$. This relation shows, by Theorem 2.1, that (2.10) is valid.

Now (2.12) is nothing but

$$\phi'(t) = \Phi^{-1} \left(t, t_0, \phi(t) \right) R \left(t, G^{-1} \left\{ G(\phi(t)) + \int_{t_0}^t \lambda(s) ds \right\} \right)$$

which yields (2.11). ||

Remark. The result (2.11) in Corollary 2.2 is a result in [2]. Here we have obtained it as a consequence of our Theorem 2.1. The other relation (2.10) is new which follows from Theorem 2.2.

A known nonlinear variation of constants type result due to Alekseev [1, or page 74 in 3] for the system (2.2) is of the form

$$(2.13) \quad y(t, t_0, x_0) = x(t, t_0, x_0) + \int_{t_0}^t \Phi(t, s, y(s)) F(s, y(s)) ds.$$

This is obtained under the assumption that $f_x(t, x)$ exists and is continuous

$$G(x(t, t_0, x_0)) = G(x_0) + \int_{t_0}^t \lambda(s) ds.$$

Hence $\frac{\partial G}{\partial x} \left(x(t, t_0, x_0) \right) \frac{\partial x}{\partial x_0} (t, t_0, x_0) = \frac{\partial G}{\partial x_0} (x_0)$, and this implies by

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This is obtained under the assumption that $f_x(t, x)$ exists and is continuous

for $(t, x) \in R^+ \times R^n$. The proof of (2.13) uses the facts that, in view of the assumptions on $f_x(t, x)$, solutions $x(t, t_0, x_0)$ of (2.1) are unique, differentiable with respect to initial data and satisfy

$$(2.14) \quad \frac{\partial x}{\partial x_0}(t, t_0, x_0) = \Phi(t, t_0, x_0),$$

$$\frac{\partial x}{\partial t_0}(t, t_0, x_0) = -\Phi(t, t_0, x_0)f(t_0, x_0),$$

where $\Phi(t, t_0, x_0)$ is the fundamental matrix solution of the variational equation

$$z' = f_x(t, x(t, t_0, x_0))z, \quad \Phi(t_0, t_0, x_0) = I.$$

Theorem 2.3. Assume that $f_x(t, x)$ exists and is continuous for $(t, x) \in R^+ \times R^n$, then formulas (2.7) and (2.13) are equivalent.

Proof. We first note that

$$(2.15) \quad x(t, s, y(s)) = x(t, t_0, \phi(s)),$$

where $\phi(t)$ is a solution of (2.5). Then differentiating with respect to s , we get

$$-\Phi(t, s, y(s))f(s, y(s)) + \Phi(t, s, y(s))y'(s) = \Phi(t, t_0, \phi(s))\phi'(s).$$

This implies substituting for y' and ϕ'

$$\Phi(t, s, y(s))F(\hat{s}, y(s)) = \Phi(t, t_0, \phi(s))\Phi^{-1}(s, t_0, \phi(s))F(s, y(s))$$

and consequently,

$$\Phi(t, s, y(s)) \equiv \Phi(t, t_0, \phi(s)) \Phi^{-1}(s, t_0, \phi(s)).$$

Hence the proof. ||

Remark. Corollary 2.2 implies that the assumptions of Theorem 2.1 and Theorem 2.2 are satisfied even when $f(t, x)$ is not differentiable with respect to x . On the other hand, Theorem 2.3 shows that if $f_x(t, x)$ is assumed to exist and is continuous, then Alekseev's formula is equivalent to our formula (2.7). Furthermore, we have provided another formula (2.3). These considerations show that our approach to a variation of parameters formula is more general and flexible.

3. Applications. Here we shall give some applications of our results. We begin with a simple use of the formula (2.3).

Theorem 3.1. In addition to the assumptions of Theorem 2.1, let us suppose that

$$i) \quad || \Phi^{-1}(t, t_0, x_0) F(t, x(t, t_0, x_0)) || \leq g(t, ||x_0||)$$

where $g \in C[R^+ \times R^+, R^+]$, $g(t, 0) \equiv 0$ and the trivial solution of

$$(3.1) \quad u' = g(t, u), \quad u(t) = u \geq 0 \text{ is uniformly stable,}$$

ii) the trivial solution of (2.1) is uniformly stable.

Then the trivial solution of (2.2) is uniformly stable.

Proof. By Theorem 2.1 any solution $y(t, t_0, x_0)$ of (2.2) satisfies

$$(3.2) \quad y(t, t_0, x_0) = x(t, t_0, \phi(t))$$

where $\phi(t)$ is a solution of (2.5). The assumption (i) then implies, setting $m(t) = ||\phi(t)||$, the inequality

$$D^+m(t) \leq g(t, m(t)),$$

which yields by the comparison theorem [Th. 1.4.1, 3] the estimate

$$(3.3) \quad ||\phi(t)|| = m(t) \leq r(t, t_0, ||x_0||), \quad t \geq t_0,$$

where $r(t, t_0, u_0)$ is the maximal solution of (3.1).

We have from (3.2)

$$(3.4) \quad ||y(t, t_0, x_0)|| = ||x(t, t_0, \phi(t))||.$$

By assumption (ii), given $\epsilon > 0$, $t_0 \in R^+$ there exists a $\delta_1(\epsilon) > 0$ such that

$$||x(t, t_0, x_0)|| < \epsilon, \quad t \geq t_0, \quad \text{if } ||x_0|| < \delta_1(\epsilon).$$

By the fact that $u \equiv 0$ of (3.1) is uniformly stable, we have, given

$\delta_1(\epsilon) > 0$, $t_0 \in R^+$, there exists a $\delta(\epsilon) > 0$ such that $r(t, t_0, u_0) < \delta_1(\epsilon)$, $t \geq t_0$, if $0 \leq u_0 < \delta(\epsilon)$. Thus the relations (3.3) and (3.4) yield that $||x_0|| < \delta(\epsilon)$ implies $||y(t, t_0, x_0)|| < \epsilon$, $t \geq t_0$. This completes the proof.

The next application uses the formula (2.7).

Theorem 3.2. In addition to the assumptions of Theorem 2.1, suppose that

$$(i) \quad ||\Phi(t, t_0, x_0)|| \leq K \quad \text{and} \quad ||\Phi^{-1}(t, t_0, x_0)|| \leq K, \quad t \geq t_0,$$

provided $||x_0|| \leq \gamma$,

$$(ii) \quad ||F(t, y)|| \leq g(t, ||y||), \quad \text{where} \quad g \in C[R^+ \times R^+, R^+], \quad g(t, 0) \equiv 0,$$

$g(t, u)$ is non-decreasing in u , and the trivial solution of

$$u' = K^2 g(t, u), \quad u(t_0) = u_0 \geq 0,$$

is stable.

Then the trivial solution of (2.2) is stable.

Proof. The solution $x(t, t_0, x_0)$ of (2.1) is related to $\Phi(t, t_0, x_0)$ by

$$x(t, t_0, x_0) = \left(\int_0^1 \Phi(t, t_0, sx_0) ds \right) \cdot x_0,$$

as can be seen by integrating

$$\frac{dx}{ds}(t, t_0, sx_0) = \frac{\partial x}{\partial x_0}(t, t_0, sx_0) \cdot x_0 = \Phi(t, t_0, sx_0) \cdot x_0$$

from $s = 0$ to $s = 1$. Therefore, assumption (i) implies

$$||x(t, t_0, x_0)|| \leq K ||x_0||, \quad \text{for} \quad ||x_0|| \leq \gamma.$$

Now suppose there exists a $t_1 > t_0$ such that $||\phi(t)|| < \gamma$ for

$t_0 \leq t < t_1$ and $||\phi(t_1)|| = \gamma$. Then for $t_0 \leq t \leq t_1$, it follows from (2.5)

that

$$\begin{aligned} ||\phi(t)|| &\leq ||x_0|| + K \int_{t_0}^t g(s, ||x(s, t_0, \phi(s))||) ds \\ &\leq ||x_0|| + K \int_{t_0}^t g(s, K||\phi(s)||) ds. \end{aligned}$$

Thus

$$K||\phi(t)|| \leq K||x_0|| + K^2 \int_{t_0}^t g(s, K||\phi(s)||) ds$$

and as a consequence of Theorem 1.9.2 in [3], we have

$$K||\phi(t)|| \leq r(t, t_0, u_0), \quad t_0 \leq t \leq t_1.$$

where $r(t, t_0, u_0)$ is the maximal solution of

$$u' = K^2 g(t, u), \quad u(t_0) = u_0 \geq 0,$$

such that $K||x_0|| \leq u_0$.

By assumption (ii) given $\gamma K > 0$ there exists a $\delta_1 > 0$ such that $r(t, t_0, u_0) < \gamma K$, $t \geq t_0$, whenever $u_0 < \delta_1$. Thus if $||x_0|| < \delta_1/K$ then

$||\phi(t)|| < \gamma$ for all $t \geq t_0$. This contradicts $||\phi(t_1)|| = \gamma$. Thus

$||\phi(t)|| < \gamma$, $t \geq t_0$. Now (2.7) implies

$$||y(t, t_0, x_0)|| \leq K||x_0|| + K^2 \int_{t_0}^t g(s, ||y(s)||) ds.$$

Thus

$$||y(t, t_0, x_0)|| \leq r(t, t_0, u_0), \text{ where } r(t, t_0, u_0)$$

is the maximal solution of

$$u' = K^2 g(t, u), \quad u(t_0) = K ||x_0||.$$

Given $\epsilon > 0$ there exists a $\delta(\epsilon, t_0)$ such that $||x_0|| < \delta$ implies $r(t, t_0, K ||x_0||) < \epsilon$, $t \geq t_0$. Therefore, the null solution of (2.2) is stable. ||

The next theorem considers asymptotic equilibrium properties of solutions of (2.2).

Theorem 3.3. In addition to the assumptions of Theorem 2.1, suppose the following estimate holds

$$(3.5) \quad ||\Phi^{-1}(t, t_0, x_0) F(t, x(t, t_0, x_0))|| < g(t, ||x_0||),$$

where $g \in C[R^+ \times R^n, R^+]$ and monotone non-decreasing in u for each $t \in R^+$. Further, assume that all solutions $u(t)$ of

$$u' = g(t, u), \quad u(t_0) = u_0 > 0$$

are bounded on $[t_0, +\infty)$. Then any solution $y(t, t_0, x_0)$ of (2.2) exists for all $t \geq t_0$ and

$$(3.6) \quad \lim_{t \rightarrow \infty} x(t_0, t, y(t, t_0, x_0)) = \xi$$

exists for any starting value $(t_0, x_0) \in R^+ \times R^n$. Conversely, for any $\xi \in R^n$ there corresponds a starting value $(t_0, x_0) \in R^+ \times R^n$ and $y(t, t_0, x_0)$ a solution of (2.2) satisfying (3.6).

Proof. Any solution of (2.2) satisfies (2.4) where

$$\phi(t, t_0, x_0) = x_0 + \int_{t_0}^t \phi^{-1}(s, t_0, \phi(s)) F(s, x(s, t_0, \phi(s))) ds.$$

The estimate (3.5) guarantees $\phi(t)$ has asymptotic equilibrium by a result in [Th. 2.9.1, 3].

If the expression (2.4) is solved for $\phi(t)$ we obtain $\phi(t) = x(t_0, t, y(t))$. The conclusions of the theorem are then immediate.

Remark. If $f(t, x) = A(t)x$, where $A(t) \in C[R^+, R^n \times R^n]$ then solutions of (2.1) are given by $x(t, t_0, x_0) = X(t, t_0)x_0$ where $X(t, t_0)$ is the fundamental matrix solution of (2.1). Thus the limit in (3.6) reduces to

$$\lim_{t \rightarrow \infty} X^{-1}(t, t_0) y(t, t_0, x_0) = \xi,$$

which is a well-known result (see, e.g., Corollary 2.9.2.[3]).

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