Simulating non-normal distributions with specified L -moments and L -correlations

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This paper derives a procedure for simulating continuous non-normal distributions with specified L -moments and L -correlations in the context of power method polynomials of order three. It is demonstrated that the proposed procedure has computational advantages over the traditional product-moment procedure in terms of solving for intermediate correlations. Simulation results also demonstrate that the proposed L -moment-based procedure is an attractive alternative to the traditional procedure when distributions with more severe departures from normality are considered. Specifically, estimates of L -skew and L -kurtosis are superior to the conventional estimates of skew and kurtosis in terms of both relative bias and relative standard error. Further, the L -correlation also demonstrated to be less biased and more stable than the Pearson correlation. It is also shown how the proposed L -moment-based procedure can be extended to the larger class of power method distributions associated with polynomials of order five.

Keywords and Phrases: intermediate correlation, Monte Carlo, multivariate, NORTA, power method, pseudo-random numbers, simulation.

1 Introduction

The power method (PM) polynomial transformation is a traditional moment-matching technique used for simulating univariate or multivariate non-normal distributions (e.g. FLEISHMAN, 1978; VALE and MAURELLI, 1983; HEADRICK and SAWILOWSKY, 1999; HEADRICK, 2002; HEADRICK, 2010a). The PM has been used in studies that have included such topics as: asset pricing theories (AFFLECK-GRAVES and McDONALD, 1989), microarray analysis (POWELL, *et al.*, 2002), multivariate analysis (STEYN, 1993), price risk (MAHUL, 2003), regression (HEADRICK and ROTOU, 2001), structural equation models (BENTLER, 2004; HENSON, REISE, and KIM, 2007), and toxicology research (HOTHORN and LEHMACHER, 2007).

Although the traditional PM is often used, it also has the limitations associated with conventional moments insofar as estimates of skew, kurtosis, and other higher-order

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moments that can be substantially biased, have high variance, or can be influenced by outliers. However, these limitations were addressed by HEADRICK (2010b, 2011) where univariate PM systems were derived in the context of *L*-moment theory (HOSKING, 1990). Some of the advantages that these PM systems have are that *L*-moment estimators are (i) nearly unbiased for any sample size and have smaller variance, (ii) robust in the presence of outliers, (iii) superior in the context of distribution fitting when compared to their conventional moment-based PM counterparts.

There are alternative conventional moment-matching techniques to the PM transformation that can be used for simulating univariate or multivariate non-normal distributions. Some examples include transformations based on: the generalized lambda distributions (GLD) (RAMBERG and SCHMEISER, 1972, 1974; HEADRICK and MUGDADI, 2006; KARIAN and DUDEWICZ, 2011), the Tukey *g*-and-*h* distributions (GH) (TUKEY, 1977; MARTINEZ and IGLEWICZ, 1984; HEADRICK, KOWALCHUCK, and SHENG, 2008; KOWALCHUK and HEADRICK, 2010; HEADRICK, 2010; McDONALD and TURLEY, 2011), and the Burr Type III and Type XII distributions (BR) (BURR, 1942, 1973; TADIKAMALLA, 1980; HEADRICK, PANT, and SHENG, 2010). These transformations also have the same limitations associated with conventional moments as described above in the context of the PM. We would also note that the univariate GLD has been characterized by *L*-moments to obviate these limitations (ASQUITH, 2007; KARVANEN and NUUTINEN, 2008).

In terms of simulating multivariate non-normal distributions with controlled Pearson correlation structures (or Gaussian copulas), the PM, GLD, GH, and BR procedures all make use of the popular NORTA (NORmal To Anything) or NATAF (1962) approach. That is, the procedures all begin with generating multivariate standard normal deviates prior to transformation. However, there are two basic limitations of concern here with respect to NORTA. The first limitation arises because the Pearson correlation is not invariant under nonlinear strictly increasing transformations. This is a concern because the PM, GLD, GH, and BR transformations all have this characteristic associated with their respective quantile function(s) (e.g. HEADRICK, 2010a, Eq. 1.1, pp. 3–4; HEADRICK *et al.*, 2010, Eq. 9). Thus, the initial multivariate normal correlation structure used in the NORTA approach will not be maintained subsequent to any of the transformations. As such, the NORTA procedure must begin with the computation of an *intermediate* correlation matrix, which is different from the specified correlation matrix between the non-normal distributions. The purpose of an intermediate correlation matrix is to adjust for the non-normalization effect of a transformation such that the PM, GLD, GH, or BR procedure can generate non-normal distributions with a specified correlation matrix.

The PM has a comparative advantage among the four competing procedures in terms of computing intermediate correlations. This is because there is a straightforward equation that can be used to directly solve for all pairwise intermediate correlations (see VALE and MAURELLI, 1983, Eq. 11; HEADRICK and SAWIŁOWSKY, 1999, Eq. 7; HEADRICK, 2002, Eq. 26; HEADRICK, 2010a, Eqs. 2.59, 2.60). In contrast, the GLD, GH, and BR procedures all have the disadvantage of requiring

numerical integration techniques to solve for the intermediate correlations (HEADRICK and MUGDADI, 2006; KOWALCHUCK and HEADRICK, 2010; HEADRICK *et al.*, 2010; HEADRICK, 2010a, p.148). Further, the GLD and BR procedures also require the additional step of transforming the initial multivariate standard normal deviates to zero-one uniform deviates prior to transformation, which can be computationally expensive.

The second limitation of concern with the NORTA approach is that the absolute values of the solved intermediate correlations must be greater than (or equal to) their associated specified Pearson correlations (VORECHOVSKY and NOVAK, 2009). As such, the primary consequences of this limitation are that solved intermediate correlations may neither (i) exist in the range of $[-1, +1]$ nor (ii) yield a positive definite intermediate correlation matrix. To demonstrate, Table 1 gives two solved intermediate correlation matrices based on four PM distributions (see Figure 1) that have been used in a number of studies (e.g. HARWELL and SERLIN, 1988, 1989; HEADRICK and SAWILOWSKY, 1999, 2000; BERKOVITS, HANCOCK and NEVITT, 2000; ENDERS, 2001; OLSSON, FOSS and TROYE, 2003). Inspection of Table 1 indicates that the two intermediate correlation matrices are invalid because two values in matrix A exceed +1 and matrix B is not positive definite albeit the associated specified correlation matrix is positive definite.

It should also be pointed out that this particular limitation can be more problematic for the GLD, GH, and BR procedures. This is because functions performing numerical integration can often fail to converge or yield inadequate (or incorrect) solutions to intermediate correlations when non-normal distributions with more severe departures from normality (e.g. heavy-tailed distributions) are used.

In view of the above, the present aim is to extend the advantages of the *L*-moment-based PM (HEADRICK, 2011) from univariate to multivariate non-normal data generation. Specifically, the purpose of this study is to develop the methodology and a procedure for simulating non-normal PM distributions with specified *L*-moments and controlled *L*-correlations. The focus is on the popular standard normal-based

Table 1. Two *invalid* conventional moment-based PM intermediate correlation matrices for the distributions in Figure 1. The specified Pearson correlation matrices are in parentheses and are both positive definite. Matrix A consists of intermediate correlations that exceed +1 and Matrix B is not positive definite

	1	2	3	4
A				
1	1			
2	0.979 (0.90)	1		
3	0.980 (0.90)	0.929 (0.90)	1	
4	1.039 (0.90)	1.045 (0.90)	0.971 (0.90)	1
B				
1	1			
2	0.715 (0.60)	1		
3	0.796 (0.70)	0.566 (0.50)	1	
4	0.981 (0.85)	0.813 (0.70)	0.540 (0.50)	1

$$\begin{array}{ll} 1 & \alpha_3 = 0 \\ & \alpha_4 = 25 \end{array} \quad \begin{array}{l} \tau_3 = 0 \\ \tau_4 = 0.4225 \end{array}$$

$$\begin{array}{ll} c_1 = 0.0 & c_1 = 0.0 \\ c_2 = 0.2553 & c_2 = 0.3338 \\ c_3 = 0.0 & c_3 = 0.0 \\ c_4 = 0.2038 & c_4 = 0.2665 \end{array}$$

$$\begin{array}{ll} 2 & \alpha_3 = 3 \\ & \alpha_4 = 21 \end{array} \quad \begin{array}{l} \tau_3 = 0.3130 \\ \tau_4 = 0.3335 \end{array}$$

$$\begin{array}{ll} c_1 = -0.2523 & c_1 = -0.3203 \\ c_2 = 0.4186 & c_2 = 0.5315 \\ c_3 = 0.2523 & c_3 = 0.3203 \\ c_4 = 0.1476 & c_4 = 0.1874 \end{array}$$

$$\begin{array}{ll} 3 & \alpha_3 = 2 \\ & \alpha_4 = 10 \end{array} \quad \begin{array}{l} \tau_3 = 0.2266 \\ \tau_4 = 0.2493 \end{array}$$

$$\begin{array}{ll} c_1 = -0.2034 & c_1 = -0.2319 \\ c_2 = 0.6304 & c_2 = 0.7185 \\ c_3 = 0.2034 & c_3 = 0.2319 \\ c_4 = 0.0987 & c_4 = 0.1126 \end{array}$$

$$\begin{array}{ll} 4 & \alpha_3 = 0 \\ & \alpha_4 = 0 \end{array} \quad \begin{array}{l} \tau_3 = 0.0 \\ \tau_4 = 0.1226 \end{array}$$

$$\begin{array}{ll} c_1 = 0.0 & c_1 = 0.0 \\ c_2 = 1.0 & c_2 = 1.0 \\ c_3 = 0.0 & c_3 = 0.0 \\ c_4 = 0.0 & c_4 = 0.0 \end{array}$$

Fig. 1. Four power method pdfs with conventional and L -moment parameters of skew (α_3) and L -skew (τ_3), kurtosis (α_4) and L -kurtosis (τ_4), and their corresponding polynomial coefficients (c_i) for Equation 3.

Table 2. Two *valid* (positive definite) *L*-moment-based PM intermediate correlation matrices for the distributions in Figure 1. The specified *L*-correlations in parentheses correspond to the same specified Pearson correlations in Table 1

	1	2	3	4
A				
1	1			
2	0.872 (0.90)	1		
3	0.872 (0.90)	0.882 (0.90)	1	
4	0.872 (0.90)	0.882 (0.90)	0.890 (0.90)	1
B				
1	1			
2	0.549 (0.60)	1		
3	0.650 (0.70)	0.466 (0.50)	1	
4	0.813 (0.85)	0.665 (0.70)	0.479 (0.50)	1

PM polynomial of order three. Some advantages of the proposed *L*-moment-based procedure are that intermediate correlations (i) can be solved directly with a single equation i.e. numerical integration is not required, (ii) cannot exist outside the range of $[-1, +1]$ as it is shown that the absolute value of an intermediate correlation will be less than (or equal to) its associated specified *L*-correlation, (iii) are generally in closer proximity to their respective specified correlations (which comprise a positive definite matrix) and thus, less likely than the traditional PM procedure to produce an intermediate correlation matrix that is not positive definite. To demonstrate, inspection of Table 2 indicates that the problems associated with the intermediate correlation matrices in Table 1 are circumvented by the new *L*-moment-based procedure. More specifically, the two solved intermediate *L*-correlation matrices in Table 2 all have values in the range of $[-1, +1]$ and are both positive definite.

The rest of the paper is organized as follows. In section 2, a summary of univariate *L*-moments and the PM is provided. In section 3, an introduction to the *L*-correlation is provided and the methodology is subsequently developed for extending the univariate *L*-moment-based PM to multivariate non-normal data generation in the context of polynomials of order three. In section 4, the steps for implementing the new *L*-moment-based PM procedure are described. A numerical example and the results of a Monte Carlo simulation are also provided to compare the new procedure with the traditional or conventional moment-based PM procedure. In section 5, the results of the simulation study are discussed and it is also shown how the proposed procedure can be extended to the larger class of fifth-order PM polynomials.

2 Summary of univariate *L*-moments and the power method

2.1 Univariate *L*-moments

The system of *L*-moments (HOSKING, 1990, 1992, 2007; HOSKING and WALLIS, 1997) can be considered in terms of the expectations of linear combinations of order statistics associated with a random variable *X*. Specifically, the first four *L*-moments are expressed as

$$\lambda_1 = E[X_{1:1}]$$

$$\lambda_2 = \frac{1}{2} E[X_{2:2} - X_{1:2}]$$

$$\lambda_3 = \frac{1}{3} E[X_{3:3} - 2X_{2:3} + X_{1:3}]$$

$$\lambda_4 = \frac{1}{4} E[X_{4:4} - 3X_{3:4} + 3X_{2:4} - X_{1:4}]$$

or more generally as

$$\lambda_r = \frac{1}{r} \sum_{j=0}^{r-1} (-1)^j \binom{r-1}{j} E[X_{r-j:r}] \quad (1)$$

where the order statistics $X_{1:n} \leq X_{2:n} \leq \dots \leq X_{n:n}$ are drawn from the random variable X . The values of λ_1 and λ_2 are measures of location and scale and are the arithmetic mean and one-half the coefficient of mean difference (or Gini's index of spread), respectively. Higher order L -moments are transformed to dimensionless quantities referred to as L -moment ratios defined as $\tau_r = \lambda_r / \lambda_2$ for $r \geq 3$, and where τ_3 and τ_4 are the analogs to the conventional measures of skew and kurtosis. In general, L -moment ratios are bounded in the interval $-1 < \tau_r < 1$ as is the index of L -skew (τ_3) where a symmetric distribution implies that all L -moment ratios with odd subscripts are zero. Other smaller boundaries can be found for more specific cases. For example, the index of L -kurtosis (τ_4) has the boundary condition for continuous distributions of (e.g. JONES, 2004)

$$\frac{5\tau_3^2 - 1}{4} < \tau_4 < 1. \quad (2)$$

2.2 The power method

The power method (PM) polynomial transformation considered herein can be generally expressed as (HEADRICK, 2010a, pp. 12–13)

$$p(Z) = \sum_{i=1}^m c_i Z^{i-1} \quad (3)$$

where $Z \sim iid N(0, 1)$ with standard normal pdf and cdf denoted as $\phi(z)$ and $\Phi(z)$. In order for Equation 3 to produce a valid pdf requires that the PM transformation be a strictly increasing monotone function. This requirement implies that an inverse function exists (p^{-1}). As such, the cdf associated with Equation 3 can be expressed as $F(p(z)) = \Phi(z)$ and subsequently differentiating this cdf with respect to z will yield the PM pdf as $f(p(z)) = \phi(z)/p'(z)$. We would note that the PM cdf and pdf could also be expressed as $F(y) = \Phi(z)$ and $f(y) = \phi(z)/p'(z)$, where $z = p^{-1}(y)$. The shape of a PM distribution associated with Equation 3 is contingent on the values of the coefficients (c_i). In the context of L -moment-based PM polynomials of order three

($m=4$), the c_i in Equation 3 are determined from the following system of equations (HEADRICK, 2011, Eqs. 2.14–2.17)

$$\lambda_1 = c_1 + c_3 = 0 \quad (4)$$

$$\lambda_2 = \frac{\Delta}{2} = \frac{(4c_2 + 10c_4)/(2\sqrt{\pi})}{2} = 1/\sqrt{\pi} \quad (5)$$

$$\tau_3 = \frac{2\sqrt{3}c_3}{\Delta\pi} \quad (6)$$

$$\tau_4 = \frac{20\sqrt{2}(\delta_1 c_2 + \delta_2 c_4)}{\Delta\pi^{\frac{3}{2}}} - \frac{3}{2} \quad (7)$$

where $\Delta = (4c_2 + 10c_4)/(2\sqrt{\pi})$ is the coefficient of mean difference, $\lambda_1 = 0$ and $\lambda_2 = 1/\sqrt{\pi}$ are standardized values associated with the unit normal distribution, and the constants δ_1 and δ_2 in Equation 7 are (HEADRICK, 2011, Eqs. A.1, A.2)

$$\delta_1 = \frac{3\tan^{-1}[\sqrt{2}]}{\sqrt{2}} - \frac{3\pi}{4\sqrt{2}} = 0.360451 \dots \quad (8)$$

$$\delta_2 = \frac{15\tan^{-1}[\sqrt{2}]}{2\sqrt{2}} - \frac{15\pi}{8\sqrt{2}} + \frac{1}{4} = 1.151128 \dots \quad (9)$$

Thus, given specified values of L -skew (τ_3) and L -kurtosis (τ_4) (see Figure 2), the solutions for the coefficients (c_i) in Equation 3 can be obtained by evaluating (HEADRICK, 2011, Eq. 2.18)

$$c_1 = -c_3 = -\tau_3 \sqrt{\frac{\pi}{3}} \quad (10)$$

$$c_2 = \frac{-16\delta_2 + \sqrt{2}(3 + 2\tau_4)\pi}{8(5\delta_1 - 2\delta_2)} \quad (11)$$

$$c_4 = \frac{40\delta_1 - \sqrt{2}(3 + 2\tau_4)\pi}{20(5\delta_1 - 2\delta_2)} \quad (12)$$

In general, a standardized non-normal third-order PM distribution will have a valid pdf iff $0 < c_2 < 1$, $0 < c_4 < \frac{2}{5}$, and $c_3^2 - 3c_2c_4 < 0$ (HEADRICK, 2011). Figure 1 gives examples of valid PM pdfs with their corresponding conventional and L -moment-based parameters and coefficients. The coefficients listed in Figure 1 were computed using Equations 10–12 and Headrick's Eqs. (2.18)–(2.21) (HEADRICK, 2010a, p. 15). Figure 2 gives the graph of the boundary region for valid PM pdfs in the L -skew and L -kurtosis plane.

Boundary	$ \tau_3 $	τ_4
Lower	0	$30 \tan^{-1}[\sqrt{2}]/\pi - 9 \approx .1226$
Upper	$3/\sqrt{10\pi} \approx .5352$	$30 \tan^{-1}[\sqrt{2}]/\pi - 9 + \sqrt{2}/\pi \approx .5728$

Fig. 2. Boundary region for valid third-order standard normal power method pdfs in the L -skew $|\tau_3|$ and L -kurtosis τ_4 plane (HEADRICK, 2011). Valid asymmetric pdfs exist in the region inside the graphed boundary.

3 The L -correlation and L -moment-based power method

The coefficient of L -correlation (see SERFLING and XIAO, 2007) is introduced by considering two random variables Y_j and Y_k with distribution functions $F(Y_j)$ and $F(Y_k)$, respectively.

The second L -moments of Y_j and Y_k can alternatively be expressed as

$$\lambda_2(Y_j) = 2\text{Cov}(Y_j, F(Y_j)) \tag{13}$$

$$\lambda_2(Y_k) = 2\text{Cov}(Y_k, F(Y_k)). \tag{14}$$

The second L -comoments of Y_j toward Y_k and Y_k toward Y_j are

$$\lambda_2(Y_j, Y_k) = 2\text{Cov}(Y_j, F(Y_k)) \tag{15}$$

$$\lambda_2(Y_k, Y_j) = 2\text{Cov}(Y_k, F(Y_j)). \tag{16}$$

As such, the L -correlations of Y_j toward Y_k and Y_k toward Y_j are expressed as

$$\eta_{jk} = \frac{\lambda_2(Y_j, Y_k)}{\lambda_2(Y_j)} \tag{17}$$

$$\eta_{kj} = \frac{\lambda_2(Y_k, Y_j)}{\lambda_2(Y_k)}. \tag{18}$$

The L -correlation in Equation 17 (or Equation 18) is bounded such that $-1 \leq \eta_{jk} \leq 1$ where a value of $\eta_{jk} = 1$ ($\eta_{jk} = -1$) indicates a strictly increasing (decreasing) *monotone* relationship between the two variables. In general, we would also note that $\eta_{jk} \neq \eta_{kj}$.

In the context of the L -moment-based PM, suppose it is desired to simulate a T -variate distribution from polynomials of the form in Equation 3 with a specified L -correlation matrix and where each distribution has specified L -moment ratios. Define $p(Z_j)$ and $p(Z_k)$ as in Equation 3 where Z_j and Z_k have Pearson correlation ρ_{jk} and standard normal bivariate density of

$$f_{jk} = (2\pi(1 - \rho_{jk}^2)^{\frac{1}{2}})^{-1} \exp \left\{ - (2(1 - \rho_{jk}^2))^{-1} (z_j^2 + z_k^2 - 2\rho_{jk}z_jz_k) \right\}. \quad (19)$$

Using the PM cdf, Equation 3, and Equation 17 with the denominator standardized to $\lambda_2 = 1/\sqrt{\pi}$ as in Equation 5, and Equation 19, the L -correlation of $p(Z_j)$ toward $p(Z_k)$ can be expressed as

$$\begin{aligned} \eta_{jk} &= 2\sqrt{\pi} \operatorname{Cov} \left(p(Z_j), F(p(Z_k)) \right) = 2\sqrt{\pi} \operatorname{Cov} \left(p(Z_j), \Phi(Z_k) \right) \\ &= 2\sqrt{\pi} E[p(Z_j)\Phi(Z_k)] - 2\sqrt{\pi} E[p(Z_j)]E[\Phi(Z_k)] \\ &= 2\sqrt{\pi} \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} \sum_{i=1}^m c_{ji} z_j^{i-1} \Phi(z_k) f_{jk} dz_j dz_k - 2\sqrt{\pi} E[p(Z_j)]E[\Phi(Z_k)] \end{aligned} \quad (20)$$

Setting $m=4$ in Equation 20, for third-order polynomials, and integrating yields

$$\begin{aligned} \eta_{jk} &= \left(\sqrt{\pi}(c_{j1} + c_{j3}) + \rho_{jk} \left(c_{j2} + 3c_{j4} - \frac{1}{2}c_{j4}\rho_{jk}^2 \right) \right) - 2\sqrt{\pi}(c_{j1} + c_{j3}) \left(\frac{1}{2} \right) \\ &= \sqrt{\pi}(c_{j1} + c_{j3}) - \sqrt{\pi}(c_{j1} + c_{j3}) + \rho_{jk} \left(c_{j2} + 3c_{j4} - \frac{1}{2}c_{j4}\rho_{jk}^2 \right) \end{aligned} \quad (21)$$

since, in general, $E[p(Z_j)] = \lambda_1(p(Z_j)) = c_{j1} + c_{j3}$ based on Equation 4 and $E[\Phi(Z_k)] = 1/2$. Thus, we have

$$\eta_{jk} = \rho_{jk} \left(c_{j2} + 3c_{j4} - \frac{1}{2}c_{j4}\rho_{jk}^2 \right) \quad (22)$$

and, analogously, the L -correlation of $p(Z_k)$ toward $p(Z_j)$ is expressed as

$$\eta_{kj} = \rho_{jk} \left(c_{k2} + 3c_{k4} - \frac{1}{2}c_{k4}\rho_{jk}^2 \right) \quad (23)$$

where ρ_{jk} in Equations 22 and 23 is the intermediate (Pearson) correlation. The intermediate correlation in Equation 22 is determined by substituting the specified L -correlation (η_{jk}) and the values of solved coefficients (c_j) (from Equations 10–12) into the left and right-hand sides of Equation 22, respectively, and then numerically solving for ρ_{jk} . Given ρ_{jk} from Equation 22, the L -correlation η_{kj} can be determined by evaluating Equation 23 using the values for the coefficients c_k . Note for the special case of where $c_{j2} = c_{k2}$, $c_{j4} = c_{k4}$, in Equations 22 and 23, then $\eta_{jk} = \eta_{kj}$.

REMARK 1. Inspection of Equation 22 indicates that $\eta_{jk} = \rho_{jk}$ when either: (a) $p(Z_j)$ is standard normal ($c_{j2} = 1$; $c_{j4} = 0$), (b) $\rho_{jk} = 0$, or (c) $\rho_{jk} = 1$ since $c_2 + (5/2)c_4 = 1$ as given in Equation 5.

REMARK 2. The L -correlation η_{jk} in Equation 22 does not depend on the value of L -skew (τ_3). This can be verified by inspecting Equations 11 and 12 for c_2 and c_4 , which indicates that these coefficients depend only on the value of L -kurtosis (τ_4).

REMARK 3. In terms of non-normal third-order polynomials, where it is required that $0 < c_{j2} < 1$ and $0 < c_{j4} < \frac{2}{5}$ for a valid pdf (HEADRICK, 2011), it can be shown that Equation 22 can be expressed as

$$\eta_{jk} = \rho_{jk} + \frac{u\rho_{jk}}{5} - \frac{u\rho_{jk}^3}{5} \quad (24)$$

where $u \in [0, 1]$. Equation (24) can be derived by making use of the following general expression for L -kurtosis (see Figure 2 for the upper and lower boundary limits for τ_4)

$$\tau_4 = 30 \tan^{-1}[\sqrt{2}]/\pi - 9 + \frac{u\sqrt{2}}{\pi} \quad (25)$$

where the upper (lower) limit of τ_4 in Equation 25 is obtained when $u=1(u=0)$. Substituting Equations 8, 9 and 25 into Equations 11 and 12 and simplifying yields the simple expressions for c_2 and c_4 as

$$c_2 = 1 - u \quad (26)$$

$$c_4 = \frac{2u}{5}. \quad (27)$$

Thus, substituting Equations 26 and 27 into Equation 22 and simplifying gives the expression in Equation 24.

REMARK 4. Inspection of Equation 24 indicates that we have the following inequality

$$0 \leq |\rho_{jk}| \leq |\eta_{jk}| \leq 1. \quad (28)$$

As such, and unlike the conventional moment-based PM, solved intermediate correlations (ρ_{jk}) based on Equation 22 cannot exist outside the range of $[-1, +1]$. Table 2 gives an example of applying Equation 22 using the coefficients associated with the four distributions in Figure 1. Inspection of Table 2 indicates that all intermediate L -correlations (i) are less than their respective specified L -correlation, (ii) comprise positive definite matrices, (iii) do not have the problems associated with the traditional PM intermediate correlations in Table 1.

REMARK 5. In terms of Equations 22 and 23, the maximum value of the absolute difference between η_{jk} and η_{kj} is bounded in the range of

$$\max |\eta_{jk} - \eta_{kj}| \in \left[0, \frac{2}{15\sqrt{3}} = 0.076 \dots\right]. \quad (29)$$

The interval associated with Equation 29 can be shown by analogously expressing η_{kj} as η_{jk} in Equation 24, which is as follows

$$\eta_{kj} = \rho_{jk} + \frac{v\rho_{jk}}{5} - \frac{v\rho_{jk}^3}{5}$$

where $v \in [0, 1]$. Thus, using Equation 24 with $u = 1 - c_{j2}$ and $v = 1 - c_{k2}$ based on Equation 26, we have

$$\eta_{jk} - \eta_{kj} = -\frac{1}{5}\rho_{jk}(\rho_{jk}^2 - 1)(u - v). \quad (30)$$

Differentiating Equation 30 with respect to ρ_{jk} and equating the resulting expression to zero and subsequently solving for ρ_{jk} yields a critical number of $\rho_{jk} = \pm 1/\sqrt{3}$. Substituting this critical number into Equation 30 yields

$$\max |\eta_{jk} - \eta_{kj}| = \left| \frac{2(u - v)}{15\sqrt{3}} \right| = \left| \frac{2(c_{k2} - c_{j2})}{15\sqrt{3}} \right|. \quad (31)$$

Inspection of Equation 31 indicates that Equation 29 is at its upper limit when $u = 1$ and $v = 0$ (or $u = 0, v = 1$) This implies that the j th distribution has the maximum level of $\tau_4 \approx 0.5728$ and the k th distribution has the minimum level of $\tau_4 \approx 0.1226$ (or vice-versa) as shown in Equation 25 and in Figure 2. Equation 29 is at its lower limit of zero when $u = v$. This implies that the j th and k th distributions have the same value of τ_4 because $c_{j2} = c_{k2}$ and $c_{j4} = c_{k4}$ based on Equations 7 and 27 and thus $\eta_{jk} = \eta_{kj}$ in Equations 22 and 23.

4 The procedure and simulation study

To implement the procedure for simulating non-normal PM distributions with specified L -moments and controlled L -correlations we suggest the following five steps:

1. Specify the L -moments for T polynomials of the form in Equation 3 i.e. $p(Z_1), \dots, p(Z_T)$ and obtain the coefficients $(c_1, \dots, c_4)$ for each polynomial by evaluating Equations 10–12 using the specified values of L -skew (τ_3) and L -kurtosis (τ_4) for each distribution. Specify a $T \times T$ matrix of L -correlations for $p(Z_j)$ toward $p(Z_k)$, where $j < k \in \{1, 2, \dots, T\}$.
2. Compute the intermediate Pearson correlations ρ_{jk} by substituting the specified L -correlation η_{jk} and the coefficients for c_{j2} and c_{j4} from Step 1 into the left-and the right-hand sides of Equation 22, respectively, and then numerically solve for ρ_{jk} . Repeat this step separately for all $T(T - 1)/2$ pairwise combinations of correlations.
3. Assemble the intermediate correlations ρ_{jk} from Step 2 into a $T \times T$ matrix and decompose this matrix using a Cholesky factorization. Note that this step requires the intermediate correlation matrix to be positive definite.
4. Use the results of the Cholesky factorization from Step 3 to generate T standard normal variables $(Z_1, \dots, Z_T)$ correlated at the intermediate levels as follows

$$\begin{aligned}
 Z_1 &= a_{11} V_1 \\
 Z_2 &= a_{12} V_1 + a_{22} V_2 \\
 &\vdots \\
 Z_j &= a_{1j} V_1 + a_{2j} V_2 + \cdots + a_{ij} V_i + \cdots + a_{jj} V_j \\
 &\vdots \\
 Z_T &= a_{1T} V_1 + a_{2T} V_2 + \cdots + a_{iT} V_i + \cdots + a_{jT} V_j + \cdots + a_{TT} V_T
 \end{aligned} \tag{32}$$

where $V_1, \dots, V_T$ are independent standard normal random variables and where a_{ij} represents the element in the i th row and the j th column of the matrix associated with the Cholesky factorization performed in Step 3.

5. Substitute $Z_1, \dots, Z_T$ from Step 4 into the T power method polynomials in Step 1 to generate the non-normal distributions with the specified L -moments and L -correlations.

To demonstrate the steps above, and evaluate the proposed L -moment procedure, a comparison between the new and conventional moment-based procedures is subsequently described. Specifically, the distributions in Figure 1 are used as a basis for the comparison using the specified correlation matrices in Table 3 where both strong and moderate levels of correlation are considered. Note in Figure 1 that the order of selection for the distributions in terms of τ_4 keeps the maximum absolute differences between η_{12} and η_{21} ; η_{23} and η_{32} ; η_{34} and η_{43} based on Equation 31 fairly small (0.015; 0.019, and 0.021, respectively).

Tables 4 and 5 give the solved intermediate correlation matrices for the L -moment and conventional moment-based procedures, respectively. Note that the intermediate correlations for the conventional procedure were computed using Headrick's Equation 2.60 (HEADRICK, 2010a, p. 30). Tables 6 and 7 give the results of the Cholesky decompositions on the intermediate correlation matrices, which are then used to create $Z_1, \dots, Z_4$ with the specified intermediate correlations by making use of

Table 3. Specified correlation matrices for the distributions in Figure 1

	1	2	3	4
A				
1	1			
2	0.70	1		
3	0.70	0.70	1	
4	0.85	0.70	0.70	1
B				
1	1			
2	0.40	1		
3	0.50	0.40	1	
4	0.60	0.50	0.40	1

Table 4. Intermediate correlations for the Conventional moment procedure

	1	2	3	4
A				
1	1			
2	0.809	1		
3	0.796	0.756	1	
4	0.981	0.813	0.755	1
B				
1	1			
2	0.505	1		
3	0.592	0.463	1	
4	0.692	0.580	0.432	1

Table 5. Intermediate correlations for the *L*-moment procedure

	1	2	3	4
A				
1	1			
2	0.650	1		
3	0.650	0.665	1	
4	0.813	0.665	0.679	1
B				
1	1			
2	0.358	1		
3	0.452	0.370	1	
4	0.549	0.466	0.382	1

Table 6. Cholesky decompositions for the Conventional moment procedure

A				
$a_{11}=1$	$a_{12}=0.809$	$a_{13}=0.796$	$a_{14}=0.981$	
0	$a_{22}=0.587$	$a_{23}=0.190$	$a_{24}=0.0316$	
0	0	$a_{33}=0.575$	$a_{34}=-0.0546$	
0	0	0	$a_{44}=0.184$	
B				
$a_{11}=1$	$a_{12}=0.505$	$a_{13}=0.592$	$a_{14}=0.692$	
0	$a_{22}=0.863$	$a_{23}=0.190$	$a_{24}=0.268$	
0	0	$a_{33}=0.784$	$a_{34}=-0.0368$	
0	0	0	$a_{44}=0.669$	

Table 7. Cholesky decompositions for the *L*- moment procedure

A				
$a_{11}=1$	$a_{12}=0.650$	$a_{13}=0.650$	$a_{14}=0.813$	
0	$a_{22}=0.760$	$a_{23}=0.320$	$a_{24}=0.180$	
0	0	$a_{33}=0.689$	$a_{34}=0.135$	
0	0	0	$a_{44}=0.536$	
B				
$a_{11}=1$	$a_{12}=0.358$	$a_{13}=0.452$	$a_{14}=0.549$	
0	$a_{22}=0.934$	$a_{23}=0.223$	$a_{24}=0.288$	
0	0	$a_{33}=0.864$	$a_{34}=0.0802$	
0	0	0	$a_{44}=0.781$	

the formulae given in Equation 32 of Step 4 with $T=4$. The values of $Z_1, \dots, Z_4$ are subsequently substituted into equations of the form of Equation 3 to produce $p(Z_1), \dots, p(Z_4)$ for both procedures.

In terms of the simulation, a Fortran algorithm was written to generate 25,000 independent sample estimates of (i) skew and kurtosis ($\alpha_{3,4}$), (ii) L -skew and L -kurtosis ($\tau_{3,4}$), (iii) the Pearson and L -correlations (ρ_{jk}^*, η_{jk}) based on samples of sizes $n=25$ and $n=1000$. The estimates for $\alpha_{3,4}$ were based on Fisher's k -statistics i.e. the formulae currently used by most commercial software packages such as SAS, SPSS, Minitab, etc. for computing indices of skew and kurtosis (where $\alpha_{3,4}=0$ for the standard normal distribution). The formulae used for computing estimates for $\tau_{3,4}$ were Headrick's Eqs 2.4 and 2.6 (HEADRICK, 2011). The estimates for ρ_{jk}^* were based on the usual formula for the Pearson product-moment of correlation statistic and the estimate for η_{jk} was computed based on Equation 17 using the empirical forms of the cdfs in Equations 13 and 15. The estimates of ρ_{jk}^* and η_{jk} were both transformed using Fisher's z' transformation. Bias-corrected accelerated bootstrapped average (mean) estimates, confidence intervals (C.I.s), and standard errors were subsequently obtained for the estimates associated with the parameters ($\alpha_{3,4}, \tau_{3,4}, z'_{\rho_{jk}}, z'_{\eta_{jk}}$) using 10,000 resamples via the commercial software package Spotfire S+ (TIBCO, 2008). The bootstrap results for the estimates of the means and C.I.s associated with $z'_{\rho_{jk}}$ and $z'_{\eta_{jk}}$ were then transformed back to their original metrics (i.e. estimates for ρ_{jk}^*, η_{jk}). Further, if a parameter (P) was outside its associated bootstrap C.I., then an index of relative bias (RB) was computed for the estimate (E) as: $RB = ((E-P)/P) \times 100$. Note that the small amount of bias associated with any bootstrap C.I. containing a parameter was considered negligible and thus not reported. The results of the simulation are reported in Tables 8–13 and are discussed in section 5.

Table 8. Skew (α_3) and Kurtosis (α_4) results for the Conventional moment procedure

Dist	Parameter	Estimate	95% Bootstrap C.I.	Standard Error	Relative Bias %
A. $n=25$					
1	$\alpha_3=0$	0.019	−0.004, 0.041	0.0117	—
	$\alpha_4=25$	5.75	5.69, 5.81	0.0293	−77.0
2	$\alpha_3=3$	1.56	1.54, 1.57	0.0077	−48.0
	$\alpha_4=21$	5.03	4.97, 5.09	0.0291	−76.0
3	$\alpha_3=2$	1.17	1.16, 1.18	0.0062	−41.5
	$\alpha_4=10$	3.06	3.01, 3.10	0.0227	−69.4
4	$\alpha_3=0$	0.0038	−0.0019, 0.0095	0.0029	—
	$\alpha_4=0$	−0.0014	−0.0121, 0.0104	0.0057	—
B. $n=1000$					
1	$\alpha_3=0$	−0.002	−0.021, 0.015	0.0090	—
	$\alpha_4=25$	21.3	21.1, 21.5	0.0948	−14.8
2	$\alpha_3=3$	2.85	2.84, 2.86	0.0053	−5.00
	$\alpha_4=21$	18.2	18.1, 18.4	0.0774	−13.3
3	$\alpha_3=2$	1.94	1.93, 1.95	0.0032	−3.00
	$\alpha_4=10$	9.15	9.08, 9.23	0.0370	−8.50
4	$\alpha_3=0$	−0.0001	−0.0010, 0.0009	0.0005	—
	$\alpha_4=0$	−0.0006	−0.0024, 0.0014	0.0010	—

Table 9. L -skew (τ_3) and L -kurtosis (τ_4) results

Dist	Parameter	Estimate	95% Bootstrap C.I.	Standard Error	Relative Bias %
A. $n=25$					
1	$\tau_3=0$	0.0022	-0.0008, 0.0054	0.0016	—
	$\tau_4=0.4225$	0.4034	0.4019, 0.4049	0.0008	-4.52
2	$\tau_3=0.3130$	0.2930	0.2908, 0.2951	0.0011	-6.39
	$\tau_4=0.3335$	0.3202	0.3186, 0.3216	0.0008	-3.99
3	$\tau_3=0.2266$	0.2133	0.2114, 0.2150	0.0009	-5.87
	$\tau_4=0.2493$	0.2412	0.2399, 0.2425	0.0007	-3.25
4	$\tau_3=0$	0.0006	-0.0005, 0.0018	0.0006	—
	$\tau_4=0.1226$	0.1239	0.1230, 0.1248	0.0004	1.06
B. $n=1000$					
1	$\tau_3=0$	-0.0001	-0.0006, 0.0005	0.0003	—
	$\tau_4=0.4225$	0.4219	0.4217, 0.4221	0.0001	-0.14
2	$\tau_3=0.3130$	0.3124	0.3120, 0.3128	0.0002	-0.19
	$\tau_4=0.3335$	0.3331	0.3329, 0.3334	0.0001	-0.12
3	$\tau_3=0.2266$	0.2262	0.2259, 0.2265	0.0001	-0.18
	$\tau_4=0.2493$	0.2490	0.2489, 0.2493	0.0001	—
4	$\tau_3=0$	-0.0000	-0.0002, 0.0002	0.0001	—
	$\tau_4=0.1226$	0.1226	0.1225, 0.1228	0.0001	—

Table 10. Correlation (strong) results for the Conventional moment procedure

Parameter	Estimate	95% Bootstrap C.I.	Standard Error	Relative Bias%
A. $n=25$				
$\rho_{12}^*=0.70$	0.751	0.749, 0.752	0.0020	7.29
$\rho_{13}^*=0.70$	0.745	0.744, 0.747	0.0017	6.43
$\rho_{14}^*=0.85$	0.908	0.907, 0.908	0.0013	6.82
$\rho_{23}^*=0.70$	0.736	0.734, 0.737	0.0018	5.27
$\rho_{24}^*=0.70$	0.748	0.746, 0.749	0.0012	6.86
$\rho_{34}^*=0.70$	0.727	0.726, 0.728	0.0012	3.86
B. $n=1000$				
$\rho_{12}^*=0.70$	0.7025	0.7021, 0.7029	0.00039	0.357
$\rho_{13}^*=0.70$	0.7025	0.7022, 0.7029	0.00034	0.357
$\rho_{14}^*=0.85$	0.8543	0.8541, 0.8545	0.00043	0.506
$\rho_{23}^*=0.70$	0.7013	0.7009, 0.7016	0.00034	0.186
$\rho_{24}^*=0.70$	0.7025	0.7023, 0.7028	0.00024	0.357
$\rho_{34}^*=0.70$	0.7013	0.7010, 0.7015	0.00021	0.186

Table 11. Correlation (strong) results for the L -moment procedure

Parameter	Estimate	95% Bootstrap C.I.	Standard Error	Relative Bias%
A. $n=25$				
$\eta_{12}=0.70$	0.704	0.703, 0.706	0.0017	0.57
$\eta_{13}=0.70$	0.704	0.703, 0.706	0.0017	0.57
$\eta_{14}=0.85$	0.852	0.851, 0.853	0.0017	0.24
$\eta_{23}=0.70$	0.707	0.706, 0.709	0.0017	1.00
$\eta_{24}=0.70$	0.706	0.705, 0.708	0.0017	0.86
$\eta_{34}=0.70$	0.707	0.706, 0.709	0.0015	1.00
B. $n=1000$				
$\eta_{12}=0.70$	0.6999	0.6996, 0.7001	0.00026	—
$\eta_{13}=0.70$	0.7001	0.6999, 0.7004	0.00026	—
$\eta_{14}=0.85$	0.8500	0.8498, 0.8501	0.00025	—
$\eta_{23}=0.70$	0.6998	0.6996, 0.7001	0.00025	—
$\eta_{24}=0.70$	0.7001	0.6998, 0.7003	0.00025	—
$\eta_{34}=0.70$	0.7001	0.6999, 0.7003	0.00023	—

Table 12. Correlation (moderate) results for the Conventional moment procedure

Parameter	Estimate	95% Bootstrap C.I.	Standard Error	Relative Bias%
A. $n = 25$				
$\rho_{12}^* = 0.40$	0.452	0.449, 0.454	0.0017	13.0
$\rho_{13}^* = 0.50$	0.546	0.543, 0.548	0.0016	9.20
$\rho_{14}^* = 0.60$	0.643	0.642, 0.645	0.0013	7.17
$\rho_{23}^* = 0.40$	0.436	0.437, 0.439	0.0017	9.00
$\rho_{24}^* = 0.50$	0.537	0.535, 0.539	0.0013	7.40
$\rho_{34}^* = 0.40$	0.419	0.417, 0.421	0.0013	4.75
B. $n = 1000$				
$\rho_{12}^* = 0.40$	0.4023	0.4019, 0.4028	0.00029	0.575
$\rho_{13}^* = 0.50$	0.5025	0.5021, 0.5029	0.00028	0.500
$\rho_{14}^* = 0.60$	0.6025	0.6021, 0.6027	0.00022	0.417
$\rho_{23}^* = 0.40$	0.4011	0.4006, 0.4015	0.00028	0.275
$\rho_{24}^* = 0.50$	0.5018	0.5015, 0.5021	0.00020	0.360
$\rho_{34}^* = 0.40$	0.4007	0.4004, 0.4010	0.00020	0.175

Table 13. Correlation (moderate) results for the L -moment procedure

Parameter	Estimate	95% Bootstrap C.I.	Standard Error	Relative Bias%
A. $n = 25$				
$\eta_{12} = 0.40$	0.406	0.403, 0.408	0.0017	1.50
$\eta_{13} = 0.50$	0.504	0.502, 0.507	0.0017	0.80
$\eta_{14} = 0.60$	0.604	0.602, 0.606	0.0017	0.67
$\eta_{23} = 0.40$	0.406	0.403, 0.409	0.0017	1.50
$\eta_{24} = 0.50$	0.506	0.504, 0.509	0.0017	1.20
$\eta_{34} = 0.40$	0.407	0.405, 0.409	0.0015	1.75
B. $n = 1000$				
$\eta_{12} = 0.40$	0.4000	0.3996, 0.4004	0.00026	—
$\eta_{13} = 0.50$	0.5001	0.4998, 0.5005	0.00026	—
$\eta_{14} = 0.60$	0.5999	0.5996, 0.6002	0.00026	—
$\eta_{23} = 0.40$	0.3995	0.3992, 0.4000	0.00026	—
$\eta_{24} = 0.50$	0.5001	0.4998, 0.5005	0.00026	—
$\eta_{34} = 0.40$	0.4000	0.3997, 0.4004	0.00023	—

5 Discussion

One of the advantages that L -moment ratios have over conventional moment-based estimators is that they can be far less biased when sampling is from distributions with more severe departures from normality (HOSKING and WALLIS 1997; SERFLING and XIAO, 2007). And, inspection of the simulation results in Tables 8 and 9 clearly indicates that this is the case. That is, the superiority that estimates of L -moment ratios (τ_3, τ_4) have over their corresponding conventional moment-based counterparts (α_3, α_4) is obvious. For example, with samples of size $n = 25$ the estimates of skew and kurtosis for Distribution 2 were, on average, only 52% and 24% of their associated population parameters whereas the estimates of L -skew and L -kurtosis were 93.61% and 96.01% of their respective parameters. Further, and in the context of the heavy-tailed distributions, it is also evident from Tables 8 and 9 that L -skew and L -kurtosis are more

efficient estimators as their relative standard errors $RSE = (\text{standard error/estimate}) \times 100$ are considerably smaller than the conventional-moment based estimators of skew and kurtosis. For example, in terms of Distribution 2, inspection of Tables 8B and 9B indicates RSE measures of: $RSE(\hat{\alpha}_3) = 0.186\%$ and $RSE(\hat{\alpha}_4) = 0.425\%$ compared with $RSE(\hat{\tau}_3) = 0.064\%$ and $RSE(\hat{\tau}_4) = 0.030\%$. This demonstrates that L -skew and L -kurtosis have more precision because they have less variance around their estimates.

Presented in Tables 10–13 are the results associated with the Pearson and L -correlations. Overall inspection of these tables clearly indicates that the L -correlation is superior to the Pearson correlation in terms of relative bias. For example, for moderate correlations ($n = 25$) the relative bias for the two heavy-tailed distributions (i.e. Distributions 1 and 2) was 13% for the Pearson correlation compared with only 1.5% for the L -correlation. Further, for large sample sizes ($n = 1000$), the L -correlation bootstrap C.I.s contained all population parameters whereas the Pearson correlation C.I.s contained none of the parameters. It is also noted that the variability of the L -correlation is more stable than that of the Pearson correlation both within and across the different conditions.

In summary, the new L -moment based PM procedure is an attractive alternative to the traditional conventional-moment-based procedure in the context of multivariate data generation. In particular, the L -moment based procedure has distinct advantages when distributions with large departures from normality are used. These advantages were demonstrated in terms of both computing intermediate correlations (see Tables 1 and 2) and in the simulation results.

It should be pointed out that like the conventional PM, the L -moment-based PM associated with third-order polynomials is limited in terms of the possible combinations of L -skew and L -kurtosis (see Figure 2) for valid pdfs. Thus, it is worthy to point out that the L -moment-based procedure can be easily extended to the larger class of distributions for fifth-order polynomials. In this context, L -kurtosis extends from the third-order boundary of $0.1226 \leq \tau_4 \leq 0.5728$ to the (approximate) fifth-order boundary of $0 \leq \tau_4 \leq 0.8$ (see HEADRICK, 2011, Figure 5 for symmetric distributions). Note that the lower boundary of τ_4 for the third-order (fifth-order) PM is the standard normal (regular uniform) distribution. Further, this broader class of fifth-order PM distributions includes many theoretical densities such as some distributions from the Gamma, Beta, Weibull, F , and Student t distributions.

The system of equations to compute coefficients for the fifth-order polynomial are given in HEADRICK (2011, Eqs. 2.8–2.13). In terms of computing intermediate correlations, if we set $m = 6$ in Equation 20, then Equation 22 extends to

$$\eta_{jk} = \rho_{jk} \left(c_{j2} + 3c_{j4} + 15c_{j6} - \frac{1}{2}c_{j4}\rho_{jk}^2 - 5c_{j6}\rho_{jk}^2 + \frac{3}{4}c_{j6}\rho_{jk}^4 \right). \quad (33)$$

It should also be pointed out, however, that the inequality in Equation 28 does not apply to the entire family of fifth-order PM distributions. Specifically, fifth-order polynomials can have valid pdfs with solutions of $c_{j2} > 1$ for values of L -kurtosis in the range of $0 \leq \tau_4 < 0.1226$.

In this case, solutions to intermediate correlations in Equation 33 will be larger than the specified L -correlation i.e. if $c_{j2} > 1$, then Equation 28 would appear as $0 \leq |\eta_{jk}| \leq |\rho_{jk}| \leq 1$.

Finally, we would note that Mathematica (WOLFRAM RESEARCH INC., 2010) source code is available from the authors for computing polynomial coefficients, L -moment ratios, intermediate correlations, and graphing power method pdfs.

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